Abstract

This paper considers the problem of correlated random variables in distributed or decentralised multiagent systems with arbitrary peer-to-peer communication topography (including loops) and temporally sparse communication. Agents either run Kalman filters or contain graphical gaussian models for which the mean and covariance of the model variables are sufficient. Each agent carries only a partial knowledge representation about the domain and utilises other agents’ state estimates. Thus, the cross covariances of the variables held by different agents are required. However, such cross covariances may be either too expensive to maintain or are simply not available and a conservative estimate for the joint covariance matrix is the best that can be achieved. Covariance Inflation (CI) is a method for generating conservative covariance matrices when cross correlations are completely unknown. We generalise CI to a new approach, called Bounded CI (BCI), which incorporates upper and/or lower bounds on the cross correlations when they are known. The new approach is applied to multiagent data fusion and inference problems for which agents have both private and shared variables. We demonstrate how bounds on the covariances of variables held by different agents can be determined efficiently and locally without the need for inter-agent modelling. The new approach is compared to vanilla CI via a simple target tracking problem.

1 Introduction

Much research has been undertaken into decentralising and distributing graphical models (or belief networks) especially in sensor networks. Decentralised systems are robust in that a single node (or agent) can be removed without necessarily affecting the integrity of the system. A popular approach is to define the graphical model in terms of networks of agents over which the model is distributed (e.g. [6]). However, these agents are highly inflexible as their behaviour and composition are restricted by the reasoning and communication algorithms that form part of the agents architecture. Agents are either built around the network structure of the graphical model or are highly restricted as to whom they can communicate directly. In [6] agents are defined to be cliques in a junction tree whereas in [2], in order to avoid rumour propagation, Gaussian likelihoods are communicated between agents along a tree structured channel network. The chief reason for placing such restrictions on agents is to guarantee the probabilistic independence properties of messages passed through the network.

This paper considers the problem of correlated random variables in distributed or decentralised multiagent systems with arbitrary peer-to-peer communication topography (including loops) and temporally sparse communication. Agents either run Kalman filters[1] or contain graphical Gaussian models[3] for which the mean and covariance of the model variables are sufficient. Each agent carries only a partial knowledge representation about the domain and may need to utilise other agents’ state estimates. Thus, the cross covariances of the variables held by different agents are required. However, such cross covariances may be either too expensive to maintain or are simply not available and a conservative estimate for the joint covariance matrix is the best that can be achieved. The goal of this paper is to introduce a method for constructing conservative covariance matrices in multiagent systems.
Section 2 introduces Covariance Inflation (CI), a mechanism for creating conservative covariance matrices for random variables with unbounded cross correlations. Then Section 3 generalises Covariance Inflation to cases when upper and/or lower bounds on the cross correlations are known, firstly to pairs of random vectors and then for the general multi-vector case. Section 4 describes how bounds on the covariances of variables held by different agents can be determined efficiently and locally without the need for inter-agent modelling. The new approach is compared to CI via a simple target tracking problem in Section 5. Finally, Section 6 discusses the limitations of the approach.

2 Covariance Inflation

This section introduces Covariance Inflation, a mechanism for creating conservative covariance matrices with unbounded cross correlations.

**Covariance Inflation (CI):** If conservative covariance matrices \( P_{xx} \in \mathbb{R}^{(m \times m)} \) and \( P_{yy} \in \mathbb{R}^{(n \times n)} \) are known but \( P_{xy} \) is completely unknown then a conservative estimate for the joint covariance matrix for the stacked vector \((x, y)\) is:

\[
P^* = \begin{bmatrix} \frac{P_{xx}}{m} & 0 \\ 0 & \frac{P_{yy}}{n} \end{bmatrix}.
\]

This result was found by Stefan B. Williams (private communication) and is a direct extension of Covariance Intersection[5]. The parameter \( \omega \) can be chosen to minimise the overall uncertainty encoded by \( P^* \). The determinant \( \det[P^*] \) is a popular measure of uncertainty and can be shown to be a function of the dimensions of \( P_{xx} \) and \( P_{yy} \), only and \( \omega = \omega_* \) where \( \omega_* = \frac{n}{m+n} \) minimises \( \det[P^*] \). The gain in precision per variable \(^1\) using the optimal choice of \( \omega \) as opposed to using the neutral choice \( \omega = 0.5 \) is independent of \( m \) and \( n \) and is (see Figure 1),

\[
\frac{0.5}{\omega_*^2 (1-\omega_*)^2 (1-\omega_*)}.
\]

In this paper we are interested in linear inference problems of the form:

\[ w = F_x x + F_y y. \]

The Kalman filter prediction and fusion stages can be written in this form. A conservative estimate of the covariance for \( w \) can be obtained from a conservative estimate of the covariance matrix \( P^* \) for the stacked vector \((x, y)\),

\[
P_{ww}^* = F P^* F^T
\]

where \( F \) is a stacked matrix with elements \( F_x \) and \( F_y \).

3 Bounded Inflation for Covariance Matrix Pairs

Let \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^n \) be two random vectors with individual covariances \( P_{xx} \) and \( P_{yy} \) where \( x \) and \( y \) are assumed to be correlated. Their covariance \( P_{xy} \) is not explicitly known. It is only known that the covariance values \( P_{xy} \) are limited according to:

\[
[C_{min}]_{ij} \leq [P_{xy}]_{ij} \leq [C_{max}]_{ij}.
\]

The goal is to find a conservative covariance \( P^* \):

\[
P^* \geq P
\]

for all possible joint covariances \( P \) defined by:

\[
P = \begin{bmatrix} P_{xx} & P_{xy} \\ P_{xy}^T & P_{yy} \end{bmatrix}.
\]

Define \( D_{xy} \):

\[
D_{xy} = \frac{C_{max} + C_{min}}{2}
\]

and let \( S \) be,

\[
S = \operatorname{arg\,min}_{r>0} \left\{ \begin{bmatrix} r P_{xx} & P_{xy} - D_{xy} \\ P_{xy}^T & r P_{yy} \end{bmatrix} \geq 0, \quad \forall [P_{xy}]_{ij} : [C_{min}]_{ij} \leq [P_{xy}]_{ij} \leq [C_{max}]_{ij} \right\}.
\]

Then \( P^* \), where:

\[
P^* = \begin{bmatrix} (1 + KS) P_{xx} & D_{xy} \\ D_{xy}^T & (1 + \frac{S}{K}) P_{yy} \end{bmatrix}
\]

and \( K > 0 \), is a conservative covariance for \( P \). The value \( K \) is called the inflation factor.

**Theorem 3.1** \( P^* \geq P \)

**Proof** For any \( K > 0 \),

\[
\begin{bmatrix} K C_{xx} & -C_{xy} \\ -C_{xy}^T & \frac{1}{K} C_{yy} \end{bmatrix} \geq 0 \quad \iff \quad \begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy}^T & C_{yy} \end{bmatrix} \geq 0.
\]

Therefore,

\[
P^* - P \geq 0 \quad \iff \quad \begin{bmatrix} K S P_{xx} & D_{xy} - P_{xy} \\ D_{xy}^T & S P_{yy} \end{bmatrix} \geq 0 \quad \iff \quad \begin{bmatrix} S P_{xx} & P_{xy} - D_{xy} \\ D_{xy}^T & S P_{yy} \end{bmatrix} \geq 0.
\]

The RHS is true by Eqn 2. □
Figure 1: The gain in precision per variable using the optimal choice of $\omega$ as opposed to using the neutral choice $\omega = 0.5$.

**Example 1:** Figure 2 compares BCI and CI matrices $P^*$ on a simple two variable scenario with $P_{xx} = 2$, $P_{yy} = 1$ and $0.5 < P_{xy} < 0.8$.

**Example 2:** When the cross-terms are unknown and $n = m$ then we regain the familiar “double the diagonal” scaling rule for inflating covariances.

$$P^* = \begin{bmatrix} 2P_{xx} & 0 \\ 0 & 2P_{yy} \end{bmatrix}.$$

**Example 3:** When $\hat{x}$ and $\hat{y}$ are state estimates over the same state spaces, $P_{xx}$ and $P_{yy}$ are the corresponding covariance matrices and $\hat{w}$ is the fused estimate with covariance $P_{ww}$ then (assuming $D_{xy} = 0$) we have

$$P^{-1}_{ww} \hat{w} = \begin{bmatrix} P^{-1}_{xx} \hat{x} \\ P^{-1}_{yy} \hat{y} \end{bmatrix},$$

Note that when $S = 0$ we recover the familiar Kalman filter Information form and when $S = 1$ and $K = \frac{\hat{w}}{\hat{x}}$ we recover Covariance Intersection[5].

Section 5 presents a fusion example for which $D_{xy} \neq 0$.

We can choose values for $K$ to minimise the overall uncertainty encoded by the covariance matrix. Popular measures of uncertainty are the determinant $\det[P^*]$ and the trace $tr(P^*)$. The Kalman filter traditionally minimises the trace of the covariance. When $D_{xy} = 0$ in Eqn 3 then the determinant is mathematically simple:

$$\det[P^*] = (1 + KS)^m \left(1 + \frac{S}{K} \right)^n \det[P_{xx}] \det[P_{yy}].$$

It is straight-forward to show that the value for $K$ which minimises $\det[P^*]$ is a function of the dimensions of the covariances $P_{xx}$ and $P_{yy}$ only and satisfies the following quadratic equation,

$$mK^2 + S(m-n)K - nx = 0.$$  \hspace{1cm} (4)

The expression for the trace is:

$$tr(P^*) = (1 + KS)tr(P_{xx}) + \left(1 + \frac{S}{K}\right)tr(P_{yy}).$$

and it is straight-forward to show that the value for $K$ which minimises $tr(P^*)$ is,

$$K = \sqrt{\frac{tr(P_{yy})}{tr(P_{xx})}}.$$  \hspace{1cm} (5)

**3.1 Calculating Values for S**

We want to find the smallest $S$, $S = S^*$ such that:

$$\begin{bmatrix} SP_{xx} & P_{xy} - D_{xy} \\ P_{xy}^T - D_{xy}^T & SP_{yy} \end{bmatrix} \geq 0.$$  \hspace{1cm} (6)

Notionally, for any vector $v = (x, y)$ we can find the smallest value of $S$, $S = S^*_v$ for vector $v$ such that:

$$S^*_v(x^TP_{xx}x + y^TP_{yy}y) + 2x^T[P_{xy} - D_{xy}]y = 0.$$  \hspace{1cm} (6)

We may then set $S^* = \max_{v \in \mathbb{R}^{m+n}} S^*_v$.

Differentiating $S^*_v$ with respect to $x$ and then forming the inner product of the derivative with $x$, $x \cdot \frac{\partial S^*_v}{\partial x} = 0$ gives $y^TP_{yy}y = x^TP_{xx}x$. Substituting into Eqn 6 we get,

$$S^* = \max_{(x,y)} \left[ \frac{x^TP_{xx} - D_{xy}}{\sqrt{x^TP_{xx}x}} \right] \left[ \frac{y^TP_{yy}y}{\sqrt{y^TP_{yy}y}} \right].$$  \hspace{1cm} (7)
Step 1 Assign $P_{xx} = Q_{11}$, $P_{yy} = Q_{22}$ and $R_{xy} = Q_{12}$. Determine $S*_{xx}$ for these two blocks using Eqn 8 and then produce a $(m_1 + n_1) \times (m_1 + n_1)$ inflated covariance matrix $P^*_{ii,jj}$ for the $2 \times 2$ blocks in the upper diagonal of $Q$ using Eqs 3 and 4 or 5.

Step 2 Starting with $I=2$, assign $P_{xx} = P^*_{I \cup \cdots \cup I}$, $P_{yy} = Q_{I+1,I+1}$ and $R_{xy} = Q_{I+1,1 \cup \cdots \cup I}$. The diagonalising matrix $\Theta_{xx}$ is a block diagonal matrix comprising the individual diagonalising matrices for each of the blocks $Q_{ii}, 1 \leq i \leq I$. Again, determine $S*$ using Eqn 8 and then produce an inflated covariance matrix $P^*_{I \cup \cdots \cup I+1}$ for the $(I+1) \times (I+1)$ blocks in the upper diagonal of $Q$ using Eqs 3 and 4 or 5.

Step 3 Repeat Step 2 for $I$ up to $N-1$.

### 4 Coupling Vectors

How do we determine upper and lower bounds on the covariances of variable estimates from different agents? Kalman filter and graphical Gaussian model estimates comprise weighted sums over the estimates of other variables in the system (including instances of variables at different times in the case of the Kalman filter). In many cases an agent will have guaranteed private data. That is, uncommunicated observations uniquely held by an agent (obtained from its own sensor perhaps). The agent may have incorporated the new data into its state estimate and therefore the agent’s state estimate will be a weighted sum over private and shared estimates.

In general an agents estimate $\hat{x}$ of the pdf mean of a variable obtained using a Kalman filter or a graphical Gaussian model will comprise a weighted sum over the priors (e.g. initial state estimate) $\hat{x}_0$, uncertainty induced by transition kernels (e.g. process stochastic noise) $\nu$, shared observation variables $z$ and pri-
vate observation variables \(\{z'\}\). The variables \(\hat{x}_0, \{\nu\}, \{z'\}\) and \(\{z\}\) are assumed to be uncorrelated. Thus, for any two distinct agents labelled 1 and 2, their estimates \(\hat{x}_1\) and \(\hat{x}_2\) are,

\[
\begin{align*}
\hat{x}_1 &= A_1\hat{x}_0 + \sum_i B_{1i}\nu_i + \sum_i C_{1i}z_{1i} + \sum_i D_{1i}z'_{1i}, \\
\hat{x}_2 &= A_2\hat{x}_0 + \sum_j B_{2j}\nu_j + \sum_j C_{2j}z_{2j} + \sum_j D_{2j}z'_{2j},
\end{align*}
\]

respectively. The calculation of the covariance \(\text{Cov}(\hat{x}_1, \hat{x}_2)\) comprises only the shared variables:

\[
\text{Cov}(\hat{x}_1, \hat{x}_2) = \text{Cov}(x_{1,\text{shared}}, x_{2,\text{shared}})
\]

where,

\[
\begin{align*}
x_{1,\text{shared}} &= A_1\hat{x}_0 + \sum_i B_{1i}\nu_i + \sum_i C_{1i}z_{1i}, \\
x_{2,\text{shared}} &= A_2\hat{x}_0 + \sum_j B_{2j}\nu_j + \sum_j C_{2j}z_{2j}.
\end{align*}
\]

The value of \(\text{Cov}(\hat{x}_1, \hat{x}_2)\) is bounded above and below by the standard deviations \(\sigma\) of the shared components, \(|\text{Cov}(\hat{x}_1, \hat{x}_2)| \leq \sigma_{x_{1,\text{shared}}}, \sigma_{x_{2,\text{shared}}}\). Values for \(\sigma\) and can be calculated efficiently and locally by each agent. When an agent communicates a vector of state estimates the standard deviation for each state variable is also communicated. The set of standard deviations is called the coupling vector.

Tighter bounds may be obtained by general considerations of the problem domain. An example is presented in the next section.

5 Application: Multi-Agent Target Tracking

We apply the methods developed in this paper to a simple multiagent decentralised tracking problem. A dynamic process \(x_t\) is tracked by three stationary agents and each agent maintains an estimate of the state of the target using the Kalman filter. All agents have the same behaviour model of the target:

\[
x_t = x_{t-1} + 0.1 + \omega_t, \quad \omega_t \sim N(0, 0.1)
\]

and they are each able to make a measurement of the target at each time step. The agents’ observation models are:

\[
z_t = x_t + \mu_t, \quad \mu_t \sim N(0, \sigma)
\]

where \(\sigma = \{0.1, 1, 2\}\) for the three agents respectively.

By combining the observations from all agents a more accurate estimate of the state of the system can be achieved than for each agent operating independently.

Kalman filter methods (such as the Channel filter) exist for agents which are continuously in contact and are able to exchange and fuse observations synchronously. Unfortunately, when transmission constraints require that agents are only allowed to communicate sporadically then the receiving agent will have difficulty fusing communicated observations with its estimate of the current state of the system. To overcome the asynchronous communication problem the agents need to do more than just communicate their observations, they have to communicate their current state estimates. However, the agents’ estimates will be correlated due to the shared information already exchanged between them and the fact they are modelling the same stochastic process. Figure 3 shows the graphical Gaussian model for two agents in the system.

In our experiments it is assumed that agents synchronise their state estimates when they are able to communicate with each other. Since the Jacobian of the observation and process models are positive then the cross correlation for state estimates held by different agents is bounded below by zero,

\[
0 \leq P_{12} \leq \sigma_{x_{1,\text{shared}}}, \sigma_{x_{2,\text{shared}}}
\]

where, \(\sigma_{x_{1,\text{shared}}} \) and \(\sigma_{x_{2,\text{shared}}} \) are the coupling vectors for both communicating agents.

When two agents make contact their estimates \(\hat{x}_{1t}\) and \(\hat{x}_{2t}\) are fused:

\[
\hat{x}_t = (1 - \kappa)\hat{x}_{1t} + \kappa\hat{x}_{2t}
\]

where \(\kappa\) is the Kalman gain matrix. The corresponding covariance update equation is:

\[
P_t = FP^*F^T
\]

where,

\[
\begin{align*}
P^* &= \begin{bmatrix}
(1 + KS) P_{11} & \sigma_{x_{1,\text{shared}}} \\
\sigma_{x_{2,\text{shared}}} & (1 + \frac{\kappa}{K}) P_{22}
\end{bmatrix}, \\
S &= \begin{bmatrix}
\sigma_1 \sqrt{2T_{11}} \\
\sigma_2 \sqrt{2T_{22}}
\end{bmatrix}, \\
F^* &= \begin{bmatrix}
(1 - \kappa) & \kappa
\end{bmatrix}.
\end{align*}
\]

It is straight-forward to show that the expression for \(\kappa\) which minimises \(P_t\) is,

\[
\kappa = \frac{\left|P_{11} - P_{12}\right| \left|P_{11} + P_{22} - 2P_{12}\right|}{P_{12} P_{11}}.
\]

The value of \(K = 1\) is chosen to minimise \(\det[P^*]\).

In our experiments the agents take it in turns to communicate, cycling between agent 1 making contact with agent 2, then agent 2 with agent 3 and then agent 3 with agent 1. A contact takes place
each 5 time intervals. Figure 4 compares the relative performances of BCI and CI. We also compare BCI using both upper and lower cross correlation bounds the cross correlations with BCI using only the upper bound on the absolute value of $P_{12}$ (that is $|P_{12}| \leq \sigma_{x,shared}\sigma_{x,shared}$).

6 Discussion

In research to date the covariance inflation factor $K$ has been chosen to minimise either the trace or the determinant of the inflated covariance matrix. However, minimisation is not transitive and minimising the inflated matrix does not guarantee that covariance matrices resulting from further inferences are also minimal. For example, suppose we have a conservative covariance matrix $P^*$ for the stacked system $(x \ y)$ and we wish to infer an estimate for $w$ thus,

$$w = F_x x + F_y y$$

and minimise $tr(P_{ww}^*)$ where $P_{ww}^* = FP^*F^T$ and,

$$F = \begin{bmatrix} F_x & F_y \end{bmatrix}.$$  

It is straightforward to show that the value of the inflation factor $K$ in $P^*$ which minimises $tr(P_{ww}^*)$ is a function of $F$,

$$K = \sqrt{\frac{F_y P_{yy} F_y^T}{F_x P_{xx} F_x^T}}$$  \hspace{1cm} (9)

and therefore the choice of $K$ depends on the future inferences $F$. Obtaining a value for $K$ using Eqn 9 is often not trivial, even for simple examples such as our scalar tracking scenario presented in Section 5.

7 Conclusions

This paper has developed a new solution to the problem of correlated random variables in distributed or decentralised multiagent systems with arbitrary peer-to-peer communication topography (including loops) and temporally sparse communication. We have generalised Covariance Inflation to include upper and/or lower bounds on the cross correlations when they are known and we have demonstrated how bounds on the covariances of variables held by different agents can be determined efficiently and locally without the need for inter-agent modelling. Limitations of the approach and issues for further research were discussed. Specifically, the need to tailor the inflated covariance to anticipated further inferences. Finally, we have demonstrated how the new method can be applied to multiagent data fusion and inference problems for which agents have both private and shared variables.  

\begin{footnote}{It has been recently been brought to our attention that Hanebeck [4] developed a similar expression for Eqn 3. However, he only considers upper bounds on the absolute cross-correlations (i.e. cases when $D_{xy} = 0$).}
Figure 4: Ratio of state variances for different Kalman filtering approaches: BCI with both upper and lower cross correlation bounds versus both CI and BCI using only the upper bound on the absolute cross correlation.

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