A Unified Framework for Game Theoretic & Probabilistic Learning

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Project Aim

Outline

- Fictitious Play
- Bayesian Data Fusion
- Variational Learning
- Variational Nash Solution
- Variational Fictitious Play
- Parallelisms
- Examples
- Future

Fictitious Play

- 2 Players, Actions $S^1, S^2$, Payoffs $U^1, U^2$
- Each player keeps update of empirical frequencies of actions = mixed strategy of opponent $\sigma^1, \sigma^2$
- Choose best response to opponent
- E.g. MaxMin Response
  
  $s^1 = \max \left[ \sigma^2(1) \cdot U^1(1) + \sigma^2(2) \cdot U^1(2) \right]$

Fictitious Play

- Graphical Model of Fictitious Play

Bayesian Data Fusion

- Consistent fusion of observations with prior model beliefs
- Hi-dimensional integration required
  
  $P(D) = \int P(D | \theta) P(\theta) d\theta$
- Numerical Integration, or
- Bounding Integrals: Variational Learning
Variational Learning

- Convert to Convex Optimisation problem:
  \[ \max P(D) \equiv \max (\log P(D)) \]
- Cost function: KL-Divergence
  \[ \log P(D) \geq -\int Q(\theta) \log P(\theta \mid D) d\theta + H(\theta) \]

Variational Nash Solution

- Mean-Field independence assumption: specific choice that
  \[ Q(\theta) = \prod Q(\theta_i) \]
- KL Divergence minimisation by partial differentiation
  \[ \frac{\partial KL}{\partial Q(\theta)} \rightarrow \max Q(\theta) \quad \text{s.t.} \quad Q(\theta_i) = \text{const} \]
  - Nash equilibrium as minimum of function

Variational Fictitious Play

- General solution (model free): iterate
  \[ Q(\theta) = \exp \left[ \int Q(\theta_i) \log P(\theta \mid D) \right] d\theta \]
- Best Response in continuous fictitious play
  \[ Q(\theta) = \text{BR}\{Q(\theta_i)\} + Q(\theta_i) \]

Machine Learning Games - Parallels

- Cost function in games lacks entropy term
  - No trade-off between max reward and cost of play
- Best response function in games linear, in machine learning exponential
- Pay-off function is machine learning set by nature (and scientist’s model guess)

Machine Learning Games – Parallels (contd.)

- Log-probability (a.k.a. information):
  - machine learning equivalent to “currency”
- Players in machine learning “Believers” in latent parameters
  - Players’ mixed strategy is posterior parameter distributions
- Structured variational solution:
  - machine learning equivalent to “coalition”

Example: Paper-Rock-Scissors

- Modified payoff matrix gives biased solution:
  \[ M = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} \]
- Mixed Strategy solution with Linear Programming:
  - Row=[.6 0 .4]; Column=[.0 .6 .4];
Example: Paper-Rock-Scissors

- Variational Fictitious play
- Best Response adjusted for extra entropy term
- Iterate mixed strategy – i.e. no actions drawn from mixed strategy

Example: Market Allocation

- K-service providers, N-customers
- Game with continuous action spaces
- Payoff function:
  - VCG auction utility equivalent to Variational cost function when strategies are pure (Entropy = 0)
- Model complexity = optimal # of Agents

Future

- What might it buy us
  - Mean field = independent agents; Belief Propagation = coalitions
  - Fictitious play when variational solution not analytic
  - Probabilistic inference methods derived not from physics but based on human learning in games
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