Lie Groups and Algebras for optimisation and motion representation

AVL/MRG Reading Group

Tuesday 6th May 2008

- Chapter 2, An invitation to 3D vision, Ma & al
- Chapter 5, PhD Mei 2007
- Computing MAP trajectories by representing, propagating and combining PDFs over groups, Smith & al, ICCV 2003
Why use Lie Groups?

Some uses...

- Interpolation
- Motion representation
- General theory for the minimal representation of geometric objects
- Representation of PDFs over groups
Outline

1. Definitions
2. Representing motion and geometric objects
3. Interpolation
4. Minimisation
5. Uncertainty
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Matrix Lie group (1/2)

Properties of a group \((G, \circ)\) :

- closed : \((g_1, g_2) \in G^2 \Rightarrow g_1 \circ g_2 \in G,
- associative :
  \[ \forall (g_1, g_2, g_3) \in G^3, (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3), \]
- has a neutral (unit) element \(e\) :
  \[ \forall g \in G^3, e \circ g = g \circ e = g, \]
- \(\circ\) is invertible :
  \[ \forall g \in G, \exists g^{-1} \in G | g \circ g^{-1} = g^{-1} \circ g = e \]
Matrix Lie group (2/2)

Lie group \((G, \circ)\)

- \((G, \circ)\) is a group,
- \(G\) is a smooth manifold, i.e., has the topology of \(\mathbb{R}^n\), (the inverse function is differentiable everywhere)

All closed subgroups of the general linear group \(\text{GL}(n)\) (group of all invertible matrices) are Lie groups.
### Definitions

- Representing motion and geometric objects
- Interpolation
- Minimisation
- Uncertainty

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## Matrix exponential (1/2)

<table>
<thead>
<tr>
<th><strong>e^A</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>[ e^A = I_n + \sum_{p \geq 1} \frac{A^p}{p!} = \sum_{p \geq 0} \frac{A^p}{p!} , \text{ beware: } e^X e^Y \neq e^{X+Y} ]</td>
</tr>
</tbody>
</table>

This series is absolutely convergent and thus well-defined.

<table>
<thead>
<tr>
<th><strong>log A</strong></th>
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<tr>
<td>Under the condition ( | A - I | &lt; 1 ), the logarithm of ( A ) is defined as :</td>
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<tr>
<td>[ \log A = \sum_{p \geq 0} (-1)^{p+1} \frac{(A - I)^p}{p} ]</td>
</tr>
</tbody>
</table>
Matrix exponential (2/2)

Calculating $e^A$ in practice

- explicit formulas (e.g. Rodrigues’ formula for $SO(3)$). A general way of finding explicit formulas is to use the Cayley–Hamilton theorem.
- diagonalisation (not generally a good idea),
- *Nineteen dubious ways to compute the exponential of a matrix*, Moler and Loan, 1978 (or 2003)
- *The scaling and squaring method for the matrix exponential revisited*, N. Higham, 2005 (\texttt{expm} in Matlab)
Lie algebra

Lie algebra $\mathfrak{g}$ of the Lie group $G$

The set of all matrices $X$ such that $e^{tX}$ is in $G$ for all real numbers $t$.

$\mathfrak{g}$ is an algebra (vector space + ring)

- Real vector space
  - $\forall t, tX \in G$
  - $X + Y \in G$

- $[X, Y] = XY - YX \in G$ (Lie bracket)
Definitions

Lie groups and algebras

Exponential map

If $G$ is a matrix Lie group with Lie algebra $\mathfrak{g}$, then the exponential mapping for $G$ is the map:

\[ \exp : \mathfrak{g} \rightarrow G \]

In general the mapping is neither one-to-one nor onto but provides the *link* between the group and the Lie algebra.

There exists a neighborhood $\nu$ about zero in $\mathfrak{g}$ and a neighborhood $V$ of $I$ in $G$ such that $\exp : \nu \rightarrow V$ is smooth and one-to-one onto with smooth inverse.
Path-connectedness

$G$ is **path-connected** if given any two matrices $A$ and $B$ in $G$, there exists a continuous path $A(t)$, $a \leq t \leq b$, lying in $G$ with $A(a) = A$ and $A(b) = B$.

$\mathbb{SO}(n)$, $\mathbb{SL}(n)$ and $\mathbb{SE}(n)$ are connected ($\mathbb{O}(n)$ is not).
Generators

- Let $g(t_i) = \exp(t_i A_i)$ define a subgroup of $G$, then
  
  $A_i = \left. \frac{\partial g(t_i)}{\partial t_i} \right|_{t_i=0}$

  is a generator of $\mathfrak{g}$.

- The set of generators form a basis and any element $x \in \mathfrak{g}$
  can be written:

  $$A(x) = \sum_{i=1}^{n} x_i A_i$$
Special Orthogonal Group

\[
\det(e^A) = e^{\text{trace}(A)}
\]

\[
\mathbb{SO}(3) = \{ R \in \mathbb{GL}(3) | RR^\top = I, \det(R) = +1 \}
\]

- preserves orientation (not a reflexion)

Associated Lie algebra:

\[
\mathfrak{so}(3) = \{ [\omega]_\times \in \mathbb{R}^{3\times3} | \omega \in \mathbb{R}^3 \}
\]
Lie algebra representation and Euler angles

The Lie algebra representation:

\[(x_1, x_2, x_3) \mapsto \exp(x_1 [e_1] \times + x_2 [e_2] \times + x_3 [e_3] \times)\]

Euler angles:

\[(x_1, x_2, x_3) \mapsto \exp(x_1 [e_1] \times) \exp(x_2 [e_2] \times) \exp(x_3 [e_3] \times)\]
Special Euclidean Group

\[ \mathbb{SE}(3) = \left\{ \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \in \text{GL}(4) \mid R \in \text{SO}(3), t \in \mathbb{R}^3 \right\} \]

- preserves distances
- preserves orientation (not a reflexion)

Associated Lie algebra twist:

\[ \mathfrak{se}(3) = \left\{ \begin{bmatrix} \omega \times & v \\ 0 & 0 \end{bmatrix} \mid \omega, v \in \mathbb{R}^3 \right\} \]

- \( v \) is the linear velocity
- \( \omega \) is the angular velocity
Expressing velocity

Velocity of a point in homogeneous coordinates ($\mathbf{x}(t) \in \mathfrak{se}(3)$):

$$\dot{\mathbf{X}}(t) = \mathbf{x}(t)\mathbf{X}(t)$$

If $\mathbf{Y}(t) = \mathbf{T}\mathbf{X}(t)$ with $\mathbf{T} \in SE(3)$ (change of coordinates):

$$\dot{\mathbf{Y}}(t) = \mathbf{T}\mathbf{x}(t)\mathbf{T}^{-1}\mathbf{Y}(t)$$

Adjoint map on $\mathfrak{se}(3)$:

$$Ad_T : \mathfrak{se}(3) \longrightarrow \mathfrak{se}(3)$$

$$\mathbf{x} \longmapsto \mathbf{T}\mathbf{x}\mathbf{T}^{-1}$$

Adjoint representation of $\mathfrak{se}(3)$ ($e^{ad_x} = Ad_{ex}$):

$$ad_x : \mathfrak{se}(3) \longrightarrow \mathfrak{se}(3)$$

$$\mathbf{Y} \longmapsto [\mathbf{X}, \mathbf{Y}]$$
Another example: Special Linear Group

\[ \text{SL}(3) = \{ \mathbf{H} \in \text{GL}(3) \mid \det(\mathbf{H}) = +1 \} \]

- ensures an invertible matrix with a minimal amount of parameters,
- subgroups include *affine transforms* or *translations* that are directly obtained by choosing the correct generators

This representation for a homography leads to “better” results than the “standard” minimal representation:

\[
\mathbf{H} = \begin{bmatrix}
h_1 & h_2 & h_3 \\
h_4 & h_5 & h_6 \\
h_7 & h_8 & 1
\end{bmatrix}
\]
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Interpolation

Let $T_1 = e^{x_1} \in SE(3)$ and $T_2 = e^{x_2} \in SE(3)$, a smooth trajectory can be obtained as $T(x) = e^{\lambda x_1 + (1-\lambda)x_2}$ with $\lambda = 0..1$. 
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A generic minimisation problem...

Let:

\[ f : G \rightarrow \mathbb{R} \]

\[ g \mapsto f(g) \]

We want to solve, with \( \bar{f} \in \mathbb{R} \):

\[ \bar{g} = \min_g d(f(g), \bar{f}) \]

Gradient descent update:

\[ \hat{g} \leftarrow \hat{g} + g_k \]

\( \hat{g} \) has no reason to still belong to \( G \)!!! (eg. rotation)
Using Lie algebras...

\[ h : \mathbb{R}^n \rightarrow \mathfrak{g} \rightarrow \mathbb{R} \]
\[ x \mapsto G(x) \mapsto f(\hat{g} \circ e^{G(x)}) \]

The parameterisation only needs to be valid locally.
New update:
\[ \hat{g} \leftarrow \hat{g} \circ e^{G(x_k)} \]

\( \hat{g} \) is guaranteed to still belong to the group.
Important condition: the initial value and optimal value have to be path-connected (in the case of \( O \), there are two components...).
Example...

Pose estimation (Lu et al.):

$$\min_{x,t_x} \sum_{i=1}^{n} \| (I - Q_i) (R(x) R p_i + t + t_x) \|^2$$

Jacobians:

$$\nabla_x f_i = (I - Q_i) \left[ \begin{bmatrix} e_1 \end{bmatrix} \times \begin{bmatrix} e_2 \end{bmatrix} \times \begin{bmatrix} e_3 \end{bmatrix} \right]_{3\times3\times3} R p_i$$

$$\nabla_{t_x} f_i = (I - Q_i)$$

$$R_{k+1} \leftarrow R(x) R_k$$

$$t_{k+1} \leftarrow t_k + t_x$$
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Baker-Campbell-Hausdorff formula

Solution to $Z = \log(e^X e^Y)$:

$$Z = X + Y + \frac{1}{2} [X, Y] + \frac{1}{12} [X, [X, Y]] - \frac{1}{12} [Y, [X, Y]] + \ldots$$
Further reading

- **An Elementary Introduction to Groups and Representations**, Brian C. Hall.