Why study shape?

- Most structures of clinical interest have a characteristic *shape* and *anatomical location* relative to other structures.

This shows the ventricles (black), caudate nucleus, and lentiform nucleus (yellow).

The shape and relative positioning of the *typical appearance of* anatomical objects is set out in Anatomical Atlases.
Why study shape?

• Across the normal population, instances vary in size but also in shape, while retaining the “key features” of the shape
  – Shape varies statistically
• Abnormal shape variations often characterise disease
  – Learn the “normal” shape & variations on normal subjects, and be sensitive to clinically significant variants to these norms
Shape models

- What constitutes a shape model?
- How do we represent variation in such a model?
- How do we measure variation of an instance from the model?
- How can we display/represent the variation?
- How can we use a shape model to find instances of the shape, for example in images that are hard to segment?

The diversity and complexity of anatomical shapes suggests that we learn both the shape model and allowable variation
Shape models

- Point distribution, active shape, and active appearance models (Cootes & Taylor)
- Fourier snakes (Szekely)
- Active contours (Blake)
- Parametrically-deformable models (Staib & Duncan)
- ...

Point Distribution Model

Represent a shape instance by a judiciously chosen set of point (features), each of which is a $k$-dim vector. In the simplest case $p_i = (x_i, y_i)$ and $k=2$

\[ \{ p_i : i = 1 \ldots n \} \]

The $n$ feature points are stacked into a long vector of length $kn$

\[ q = [p_1, p_2, \ldots, p_n]^T \]

Assume that for each $i$ and for each of $M$ training instances of the shape that the points are in correspondence:

\[ p_{i1}^1, p_{i2}^2, \ldots, p_{iM}^M \]
Here there are $M=12$ training instances.

One point feature, the $14^{\text{th}}$, is shown, with correspondences on each instance.

$p_{14}^{1}, p_{14}^{2}, \ldots, p_{14}^{12}$

Evidently, if the number of feature points $n$ is large, and the training set size $M$ is also large, this is going to be tedious unless it can be automated…
Aligning the shape instances

The Procrustes Algorithm is used so that the sum of distances to the mean of each shape is minimised

\[ \sum_{i=1}^{M} (q_i - \overline{q})^2 \]

1. Translate each shape instance (= stacked vector) \( q_i \) so that its centre of mass is at the origin
2. Choose one example as the mean shape and rescale so that \( \overline{q} \) is unitary
3. Record the first estimate as \( q_0 \) to define the default reference frame
4. Align all instances of the shape with the current estimate of the mean shape
5. Re-estimate the mean from the aligned shapes
6. Re-scale and re-iterate the process if necessary
Modelling shape variation using PCA – Principal Components Analysis

1. Compute the mean of the data
   \[ \overline{q} = \frac{1}{M} \sum_{i=1}^{M} q_i \]

2. Compute the covariance of the data
   \[ S = \frac{1}{M-1} \sum_{i} (q_i - \overline{q})(q_i - \overline{q})^T \]

3. Compute the eigenvectors \( \mathbf{u}_i \) and eigenvalues \( \lambda_i \) of the covariance matrix, sorted in decreasing order of eigenvalue size

4. Remove the small eigenvalues, retaining “most” (eg 98%) of the variation
   
   Choose \( t \) so that
   \[ \sum_{i=1}^{t} \lambda_i \geq 0.98 \cdot \sum_{i=1}^{M} \lambda_i \]
Principal Components Analysis

We have the eigenvectors $u_i$ sorted in order so that $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_M$

and have chosen $t$ so that $\sum_{i=1}^{t} \lambda_i \geq 0.98 \times \sum_{i=1}^{M} \lambda_i$

Now define the matrix $U$ from the top $t$ eigenvectors:

$$U = [u_1 \mid u_2 \mid \ldots \mid u_t]$$

Then we approximate any shape instance $x$ by a $t$-dimensional vector $b$

$$x \approx \bar{x} + U \cdot b$$

$$b = U^T \cdot (x - \bar{x})$$
We now approximate any instance of the shape, including the training instances, by projecting onto the first $t$ eigenvectors:

$$q = \bar{q} + \sum_{i=1}^{t} b_i u_i$$

The weight vector $b$ is identified as the characteristic of this instance of the shape

$$b = [b_1, \ldots, b_t]^T$$

Varying the weights $b_i$ enables us to explore the allowable variations in the shape.
Example: hand shapes

The PDM was learned from 18 shapes, each comprising 72 points, at finger tips, finger junctions, and equally spaced along the finger sides.
Example: face shapes

Modes 1-3

Training instances
Finding a model instance in an image

We suppose that we have learned a shape model, comprising the average shape and set of $t$ modes. We are presented with a new image and try to fit the shape model to the image: this requires both that we find the appropriate weights $b$ that define the model instance and that we find the transformation from shape space to the image, to align the model instance with the image.

If the transform is $T$ and the weight vector $b$, the instance of shape $(q, u_1, ..., u_t)$ in the image is

$$q = T\left(\overline{q} + \sum_{i=1}^{t} b_i u_i\right)$$

Typically, $T$ is a similarity: translation + rotation + uniform scaling.
Finding the model pose & parameters

Suppose we have identified a set of points $Y$ in the image. Evidently, we can seek to minimise the squared distance:

$$|Y - T(\overline{q} + \sum_{i=1}^{t} b_i u_i)|^2$$

1. Initialise $b=0$
2. Generate initial model instance: $q = (\overline{q} + \sum_{i=1}^{t} b_i u_i)$
3. Find $T$ that best aligns $q$ to $Y$ (eg similarity transform)
4. Invert pose parameters, to project $y = T^{-1}(Y)$ into model frame
5. Update the model parameters: $b = U^T(y - \overline{q})$, where $U = [u_1 | ... | u_t]$
6. Repeat from step 2 until converged
Fitting a cartilage model to a knee MRI image
Active Appearance models

- The active shape model only mobilises information about the shape of an object.
- Often there is additional important information in the form of texture, shading, …
- Adding this information to the Active Shape Model gives the Active Appearance Model.
- The model learning and fitting algorithms become correspondingly more complex.
First two modes of appearance variation of the cartilage model

Best fit of knee model to MR data, given landmarks

Multi-resolution search
Variation in the (2D) cross section of ventricles
Geodesic Interpolating Splines

• Splines do not guarantee diffeomorphism

• Extension – Geodesic Interpolating Spline
  ➢ Guaranteed diffeomorphic
  ➢ Extra mathematical structure – metric

Clamped Plate Spline

Geodesic CPS
Classifying Variation

![Graph showing classification of variation using geodesic and Mahalanobis distances. The graph includes points for Classifier, Legal Examples, Illegal Examples, and Training Set. A yellow box highlights a region of interest.](image)
Landmarks and matching

• Localised features that have an associated descriptor that is sufficiently discriminating that many features can be matched uniquely ...

Which point matches which?
(a) First mammogram $I_0$

(b) Second mammogram $I_1$. 
(a) ROI's of $I_0$
(b) ROI's of $I_1$
(a) Correspondences in $I_0$

(b) Correspondences in $I_1$
Corresponding shapes are aligned using an energy function (with a suitable regulariser) that minimises the overall distance between the signature curves of the two shapes.

Note that we do not need to extract landmarks first.
Shape: original and noisy

curvature

Distance integral invariant

Local distance integral invariant

Local Area integral invariant
Distance integral invariant

Area integral invariant

(a) $\alpha = 100$.

(b) $\alpha = 30$.

(c) $\alpha = 1$.

(a) fine scale.

(b) intermediate scale.

(c) coarse scale.
Scale space of local area integral invariant for noiseless rectangle (left) and noisy (right)

Scalogram for curvature (differential invariant) left, and integral invariant (right)
Distance between shapes as a function of increasing noise
For a given noise level (here Gaussian), the shape distance remains essentially constant.
Corresponding shapes are aligned using an energy function (with a suitable regulariser) that minimises the overall distance between the signature curves of the two shapes. Note that we do not need to extract landmarks first.