The geometries of shapes

Olivier Faugeras

NeuroMathComp Team - INRIA/ENS Paris

Mike Brady Research Symposium
Oxford, September 16th - 17th
Cartan’s moving frame method

Euclidean case:

\[
\begin{align*}
\frac{\partial \mathbf{C}}{\partial s}(s, t) &= \mathbf{T}(s, t) \\
\frac{\partial \mathbf{T}}{\partial s}(s, t) &= \kappa \mathbf{N}(s, t) \\
\frac{\partial \mathbf{N}}{\partial s}(s, t) &= -\kappa \mathbf{T}(s, t)
\end{align*}
\]

$s$ is the Euclidean arc-length.
Cartan’s moving frame method

Euclidean case:

\[
\begin{align*}
\frac{\partial \mathbf{C}}{\partial s}(s, t) &= \mathbf{T}(s, t) \\
\frac{\partial \mathbf{T}}{\partial s}(s, t) &= \kappa \mathbf{N}(s, t) \\
\frac{\partial \mathbf{N}}{\partial s}(s, t) &= -\kappa \mathbf{T}(s, t)
\end{align*}
\]

\(s\) is the Euclidean arc-length.

Affine case:

\[
\begin{align*}
\frac{\partial \mathbf{C}}{\partial \sigma}(\sigma, t) &= \mathbf{T}^a(\sigma, t) \\
\frac{\partial \mathbf{T}^a}{\partial \sigma}(\sigma, t) &= \mathbf{N}^a(\sigma, t) \\
\frac{\partial \mathbf{N}^a}{\partial \sigma}(\sigma, t) &= \kappa_a \mathbf{T}^a(\sigma, t)
\end{align*}
\]

\(\sigma\) is the affine arc-length.

Olivier Faugeras

NeuroMathComp Team - INRIA/ENS Paris

The geometries of shapes
Cartan’s moving frame method
Euclidean case:

\[
\begin{aligned}
\frac{\partial \mathbf{C}}{\partial s}(s, t) &= \mathbf{T}(s, t) \\
\frac{\partial \mathbf{T}}{\partial s}(s, t) &= \kappa \mathbf{N}(s, t) \\
\frac{\partial \mathbf{N}}{\partial s}(s, t) &= -\kappa \mathbf{T}(s, t)
\end{aligned}
\]

\(s\) is the Euclidean arc-length.

Affine case:

\[
\begin{aligned}
\frac{\partial \mathbf{C}}{\partial \sigma}(\sigma, t) &= \mathbf{T}^a(\sigma, t) \\
\frac{\partial \mathbf{T}^a}{\partial \sigma}(\sigma, t) &= \mathbf{N}^a(\sigma, t) \\
\frac{\partial \mathbf{N}^a}{\partial \sigma}(\sigma, t) &= \kappa_a \mathbf{T}^a(\sigma, t)
\end{aligned}
\]

\(\sigma\) is the affine arc-length.

Shape representation in robotics and vision

Recognition
Inspection
Grasping
Reasoning
Shape scale space

The Euclidean case:

\[ \frac{\partial \mathbf{C}}{\partial t}(t, \mathbf{p}) = \kappa \mathbf{N} \quad C(0, \mathbf{p}) = \mathbf{C}_0(\mathbf{p}) \]

**Theorem (Gage and Hamilton, 1986)**

*The initial curve becomes convex in finite time and disappears as a circle of 0 radius in finite time.*
The curvature primal sketch

- Representation of subparts, *part-of*,
- of symmetries, *smoothed local symmetries*.
- Multiscale, *curvature primal sketch*.

Smoothed local symmetries

The idea:
Smoothed local symmetries

A competitor, the medial axis:
Smoothed local symmetries

Comparison smoothed local symmetry/medial axis
Smoothed local symmetries

Comparison smoothed local symmetry/medial axis
Smoothed local symmetries

Comparison smoothed local symmetry/medial axis

Olivier Faugeras
NeuroMathComp Team - INRIA/ENS Paris

The geometries of shapes
Smoothed local symmetries

Comparison smoothed local symmetry/medial axis
Smoothed local symmetries

Comparison smoothed local symmetry/medial axis
The curvature primal sketch

- Detect and make explicit important changes of curvature along the contour at several scales
- Multiscale is important because of noise.
- The detected points can be used to segment the contour, compute another approximation for smoothed local symmetries and subparts detection.
The curvature primal sketch

Five primitive curvature changes

1. Corner.
2. Smooth join.
3. End.
4. Crank.
5. Bump.
6. Dent.

and their multiscale representations are computed by convolving a Euclidean invariant representation of the boundary with the first and second order derivatives of a Gaussian.
The curvature primal sketch

Five primitive curvature changes

1. Corner.
2. Smooth join.
3. End.
4. Crank.
5. Bump.
6. Dent.

and their multiscale representations are computed by convolving a Euclidean invariant representation of the boundary with the first and second order derivatives of a Gaussian.
The curvature primal sketch

Five primitive curvature changes

1. Corner.
2. Smooth join.
3. End.
4. Crank.
5. Bump.
6. Dent.

and their multiscale representations are computed by convolving a Euclidean invariant representation of the boundary with the first and second order derivatives of a Gaussian.
Surface primal sketch

- Finding surface intersections
- Generating descriptions for the resulting smooth patches.
- Matching surface descriptions.
Surface primal sketch

The geometries of shapes
Ingredients of the surface primal sketch

- Roofs
Ingredients of the surface primal sketch

- Smooth joins

(a)

(b)
Ingredients of the surface primal sketch

- Shoulders
Surface primal sketch: principle
Surface primal sketch: example
Surface primal sketch: results
Surface primal sketch: results
Various approaches

**Random shapes:** Fréchet (tied to the parametrization) [1961], Matheron (very general) [1975], D. Kendall (tied to the point representation) [1973].


**Riemannian structure on sets of shapes:** Miller-Trouvé-Younes et al. [1998], Klassen-Srivastava [2004], Mumford-Sharon [2004]

Admissible shapes

- $\mathcal{C}$: the set of shapes whose boundary is a simple closed or open $C^2$ curve.
- $\mathcal{F}$, the set of Federer domains: the distance of the skeleton to the boundary is strictly positive.
- We work in $S = \mathcal{C} \cap \mathcal{F}_{h_0}$ (the choice of $h_0$ is guided by the grid size).
Example of admissible shapes

\[ \kappa \leq \kappa_0 = \frac{1}{h_0} \]

\[ d > h_0 \]
Deforming shapes: the topology

The Hausdorff distance:

\[ d(\Omega_1, \Omega_2) = \max(M_1 P_1, M_2 P_2) = \max(\max_{x \in \Omega_1} d(x, \Omega_2), \max_{x \in \Omega_2} d(x, \Omega_1)) \]
Deforming shapes: the topology

- Given the “smooth” energy $E : S \times S \rightarrow \mathbb{R}^+$ (continuous for one of the previous topologies) and two shapes $\Gamma$ and $\Gamma_0$.
- Given a deformation flow $\beta = \beta n$, define the Gâteaux derivative in the direction $\beta$

$$G_\Gamma(E(\Gamma, \Gamma_0), \beta) = \lim_{\varepsilon \to 0} \frac{E(\Gamma + \varepsilon \beta, \Gamma_0) - E(\Gamma, \Gamma_0)}{\varepsilon}$$
Deforming shapes: the inner product (on the tangent space)

- Given an inner product $\Pi$ on the set of normal flows, e.g.
  \[
  \langle \beta_1, \beta_2 \rangle_\Pi = \int_{\Gamma} \beta_1 \beta_2 = \int_{\Gamma} \beta_1(x)\beta_2(x) \, d\Gamma(x)
  \]

- Define the gradient of $E$ for $\Pi$
  \[
  \mathcal{G}_\Gamma(E(\Gamma, \Gamma_0), \beta) = \langle \nabla_\Pi E(\Gamma, \Gamma_0), \beta \rangle.
  \]

- Smoothly deforming $\Gamma_1$ onto $\Gamma_2$ is to solve
  \[
  \Gamma_t = -\nabla_\Pi E(\Gamma, \Gamma_2) \mathbf{n} \\
  \Gamma(0) = \Gamma_1
  \]
Deforming shapes: the inner product (on the tangent space)
Warping examples
Changing the gradient

By projecting the gradient onto a subset of possible gradients we can control the evolution and the local minimum

Hausdorff distance:

Changing the gradient

Hausdorff distance:

Rigidified Hausdorff distance:

Rigidified Hausdorff distance with landmarks:
Changing the gradient
Application: computation of the empirical covariance of a set of shapes

What is the average fish silhouette?  What is the average corpus callosum?
Application: computation of the empirical covariance of a set of shapes
It could perhaps have been done differently

Mukherjee, Zisserman, Brady, Philosophical Transactions (A), 1995
It could perhaps have been done differently

Mukherjee, Zisserman, Brady, Philosophical Transactions (A), 1995
It could perhaps have been done differently

- Exploit (Euclidean) symmetries in the objects,
- to determine image back-projection up to a similarity transformation ambiguity,
- to determine the object plane orientation (slant and tilt) and,
- as a test for non-coplanarity amongst a collection of objects.
It could perhaps have been done differently

Distinguished points

![Diagram showing distinguished points en, Tex, Ten, and h.](image-url)
It could perhaps have been done differently
It could perhaps have been done differently
The affine scale space

The affine curve evolution case:

\[ \frac{\partial C}{\partial t}(t, p) = \kappa_a N^a \quad C(0, p) = C_0(p) \]

Theorem (Sapiro and Tannenbaum, 1993)

The initial curve becomes convex in finite time and disappears as an ellipse in finite time.
The affine scale space

The affine curve evolution case:

\[ \frac{\partial \mathbf{C}}{\partial t}(t, p) = \kappa_a \mathbf{N}^a \quad C(0, p) = \mathbf{C}_0(p) \]

Theorem (Sapiro and Tannenbaum, 1993)

*The initial curve becomes convex in finite time and disappears as an ellipse in finite time.*

This result can be used to define an affine curvature primal sketch...
Hierarchical shape spaces

We are still in search of a "correct" and "workable" definition of shape spaces: Charpiat, Michor, Mumford, Sharon, Trouvé, Younes.
Hierarchical shape spaces

- We are still in search of a “correct” and “workable” definition of shape spaces: Charpiat, Michor, Mumford, Sharon, Trouvé, Younes.
- Understanding the singularities: Koenderink, Damon, Giblin.
Applications

▶ To medical images: MICCAI, MIA, . . .
Applications

- To medical images: MICCAI, MIA, . . . .
- To computer vision: ECCV, ICCV, IJCV, . . . .
Applications

- To medical images: MICCAI, MIA, . . .
- To computer vision: ECCV, ICCV, IJCV, . . .
- To computer graphics: SIGGRAPH.
Applications

- To medical images: MICCAI, MIA, . . .
- To computer vision: ECCV, ICCV, IJCV, . . .
- To computer graphics: SIGGRAPH.
- To robotics: ISRR, IJRR, . . .