1. Gaussian filter implementation

(a) A discrete approximation to a 1D Gaussian can be obtained by sampling the Gaussian function. In practice samples are taken uniformly until the values at the tails of the distribution are less that 1/1000 of the peak value of the Gaussian.

i. For $\sigma = 1$ show that the filter obtained in this way has size of 7 pixels with coefficients given by:

$$
\begin{array}{cccccc}
0.004 & 0.054 & 0.242 & 0.399 & 0.242 & 0.054 & 0.004 \\
\end{array}
$$

ii. What property of the filter coefficients ensures that regions of uniform intensity are unaffected by smoothing?

(b) The following row of pixel intensities are smoothed with the 1D Gaussian ($\sigma = 1$) given in (a).

Calculate the smoothed value of the pixel with intensity 118.

(c) Write a Matlab program to compute the smoothed values for the entire row. You may want to use the Matlab function 'conv'.

2. 1D differentiation

(a) Show that smoothing an intensity signal with a Gaussian and then differentiating the smoothed signal is equivalent to convolution with the derivative of a Gaussian: i.e. $\frac{d}{dx}[G(x)\ast I(x)] = G'(x) \ast I(x)$ where $G'(x)$ represents the first derivative of the Gaussian function.

(b) Hence, or otherwise, show how “edges” in an intensity function $I(x)$ can be localised at the zero-crossings of $G''(x) \ast I(x)$ where $G''(x)$ represents the second derivative of the Gaussian function.

(c) Suppose $I(x)$ consists of a step edge of height $h$ at $x = 0$. Derive an expression for the output of a derivative of Gaussian filter acting on $I(x)$, and show that the maximum response is at $x = 0$ and has magnitude $h/\sqrt{2\pi\sigma}$.

3. 2D spatial filters

(a) By using a Taylor expansion, determine $3 \times 3$ convolution masks for the following operations: (i) $\frac{\partial}{\partial x}$, (ii) $\frac{\partial^2}{\partial x^2}$, (iii) $\nabla^2$.

(b) By considering the Fourier transforms of a box filter and Gaussian filter, explain why an image that has been box filtered is more likely to have high frequency artifacts than one that has been Gaussian filtered.

(c) Write a Matlab program to read in an image and compare the action of a box filter to a Gaussian filter (of similar spatial scales).

4. Properties of 2D Fourier transforms

Given a function $f(x, y)$ and its 2-D Fourier transform $F(u, v)$:

(a) Derive the effect on $F$ of a general linear transformation, $x \rightarrow Ax$, where $x = (x, y)$ and $A$ is a 2D linear transformation matrix.

(b) Suppose the function $f(x, y)$ has rotational symmetry so that $f(x, y) = f(r)$ where $r^2 = x^2 + y^2$. Show that its Fourier transform is also rotationally symmetric.
5. **Gaussian filter properties and convolutions**

(a) Prove the Convolution Theorem for functions of a single variable, i.e. that the Fourier Transform of \( f(x) * g(x) \) is \( F(u)G(u) \).

(b) Show that consecutive smoothing (convolution) with a series of Gaussians,

\[
G(x) = \frac{1}{\sqrt{2\pi \sigma}} e^{\frac{-x^2}{2\sigma^2}},
\]

each with a particular standard deviation (scale) \( \sigma_j \), is equivalent to a single convolution with a Gaussian of variance \( \sum_i \sigma_i^2 \).

(c) Smoothing a 2D image involves a 2D convolution with a 2D Gaussian:

\[
G(x,y) = \frac{1}{(2\pi \sigma^2)^2} e^{\frac{-(x^2 + y^2)}{2\sigma^2}}
\]

Show that this can be performed by two 1D convolutions: i.e.

\[
G(x,y) * I(x,y) = G(x) * [G(y) * I(x,y)]
\]

What are the advantages of performing two 1D convolutions instead of a 2D convolution?

6. **Wiener filter image restoration**

(a) Suppose an image is both focus and motion blurred. The focus blur may be modelled by a Gaussian

\[
G(x,y) = \frac{1}{2\pi \sigma^2} e^{-(x^2+y^2)/2\sigma^2}
\]

and the motion blur by the product of a top-hat function in the \( x \) direction of length \( d \) and a delta function in the \( y \) direction. Write down the point spread function \( h(x,y) \) of the imaging system, and compute its Fourier transform

\[
H(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) e^{-j2\pi(u x + v y)} \, dx \, dy.
\]

(Use the result that the Fourier transform of a Gaussian \( G(x,y) \) is

\[
e^{-2\pi^2 (u^2 + v^2) \sigma^2}
\]

(b) Give the form of the Wiener filter that could be used to deblur the image.

7. **Median filter**

(a) Define what is meant by (i) a linear filter, and (ii) a separable filter. Which of these properties is satisfied by (i) a Gaussian filter, and (ii) a median filter?

(b) Suppose that salt and pepper noise is added to an otherwise uniform region of the image. Show that provided less than half of the elements in the neighbourhood are noise values, then a \( 3 \times 3 \) median filter will give the correct (i.e. uniform again) result.