European option pricing under parameter uncertainty

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Introduction
Introduction

The problem

- Stochastic volatility: parametric models, need to fit to data before employing for pricing or hedging market instruments
- Conventional approaches: (i) statistical estimation from historical asset-prices or (ii) calibration to market options
- Regardless of which approach: exposed to parameter ambiguity since point-estimates from either are subject to errors
- (i) statistical estimation: true likelihood based on distribution of asset-price $\implies$ infer confidence regions
- (ii) calibration: no likelihood but often use MSE of market-to-model prices as objective (corresponds to Gaussian observation noise)
Introduction

Approach

- In either case, assume an inferred uncertainty set for parameters (uncertainty vs. risk)

- How incorporate the effect of parameter uncertainty into option prices as outputted by the stochastic volatility model?

- Statistical estimation: establish relation to risk-neutral pricing measure and impose statistical uncertainty on risk-neutral parameters

- To avoid introducing arbitrage: fix diffusion parameters at statistical point-estimates

- Parameter uncertainty as representative for incompleteness of stochastic volatility model: exist a space of equivalent pricing measures as given by the span of risk-neutral parameters in the uncertainty set
Introduction

- We concentrate on the case (i) of statistical estimation
- Thus, can we employ the model in a consistent way with its origin, as a model for the underlying financial market with options fundamentally being derivatives, and explain the model mismatch of option market prices by introducing uncertainty?
Option pricing under parameter uncertainty
The financial market model

- We consider Heston’s stochastic volatility model for the stock price $S$
  \[ dS = \mu S dt + \sqrt{V} S (\rho dW^1 + \sqrt{1 - \rho^2} dW^2) \]
  \[ dV = \kappa (\theta - V) dt + \sigma \sqrt{V} dW^2 \]
  where $V$ is the volatility process and $\kappa$, $\theta$, $\sigma$, $\rho$ the model parameters
- The pricing measure $Q$ is usually given by the risk neutral parameters
  \[ \tilde{\kappa} = \kappa + \sigma \lambda \quad \text{and} \quad \tilde{\theta} = \frac{\kappa \theta}{\kappa + \sigma \lambda} \]
- Market price of risk parameter $\lambda$ is not endogenous to the financial market model
- No arbitrage by consistency relationships: if $\lambda$ is determined from a single exogenously given derivative, any other contingent claim is then uniquely priced
Option pricing under parameter uncertainty

- If the risk-free rate $r$ and $\kappa$, $\theta$, $\sigma$, $\rho$, $\lambda$ are fixed to some values, then Heston’s formula uniquely gives the price of a European option.
- Assume we have limited knowledge about parameters: let $\kappa$, $\theta$ and the risk-free rate $r$ lie in a compact uncertainty interval $U$.
- Introduce parameter uncertainty into the model by modifying our reference measure with the effect of a stochastic control governing the parameter processes.
Option pricing under parameter uncertainty

The risk-neutral dynamics under uncertainty

- Under $Q^u$, we have the **controlled risk-neutral dynamics** of $(S, V)$

\[
dS = r^u S dt + \sqrt{V} S (\rho dW^1 + \sqrt{1 - \rho^2} dW^2) \\
dV = \kappa^u (\theta^u - V) dt + \sigma \sqrt{V} dW^2
\]

where $u = (r^u, \kappa^u, \theta^u)$ is a **control process**, living in the uncertainty interval $U$

- Here we implicitly assume that statistical uncertainty set is the same as the uncertainty set in which our uncertain price-parameters live

- Formally, the **parameter uncertainty** is represented by the random choice of control: any $u$ among all admissible controls $\mathcal{U}$ may govern the dynamics

- Given a fixed control $u \in \mathcal{U}$, what is the price of an option?

- What are the **maximum** and **minimum** price from an optimal choice of $u \in \mathcal{U}$?
We take a look at the controlled value process

$$J_t(u) = \mathbb{E}_u \left[ e^{-\int_t^T r_s ds} g(S_T) \middle| \mathcal{F}_t \right]$$

where $E_u$ is the expectation corresponding to the controlled dynamics under $Q^u$ for a fixed $u \in \mathcal{U}$, and $g$ is the option's pay-off function.

Then, the maximum/minimum price given by

$$D_t^- = \inf_{\{u_t\} \in \mathcal{U}} \mathbb{E}_u \left[ e^{-\int_t^T r_s ds} g(S_T) \middle| \mathcal{F}_t \right],$$

$$D_t^+ = \sup_{\{u_t\} \in \mathcal{U}} \mathbb{E}_u \left[ e^{-\int_t^T r_s ds} g(S_T) \middle| \mathcal{F}_t \right]$$
Option pricing under parameter uncertainty

Key points

- Surprisingly (perhaps), we have a dual representation of $J_t(u)$ and $D_t^\pm$ by the solutions to the BSDEs

$$
\begin{align*}
    dJ_t(u) &= -f(S_t, V_t, J_t(u), Z_t, u_t)dt + Z_t d\tilde{W}_t, \\
    J_T(u) &= g(S_T), \\
    dD_t^\pm &= -H^\pm(S_t, V_t, D_t^\pm, Z_t)dt + Z_t d\tilde{W}_t, \\
    D_T^\pm &= g(S_T)
\end{align*}
$$

where these equations are linked by their driver functions

$$
\begin{align*}
    H^-(s, v, y, z) &= \inf_{u \in U} f(s, v, y, z, u), \\
    H^+(s, v, y, z) &= \sup_{u \in U} f(s, v, y, z, u)
\end{align*}
$$

- $\implies$ optimisation over functional space $\mathcal{U}$ replaced by pointwise optimisation over compact set $U$

- This representation goes back to Marie-Claire Quenez [Quenez, 1997]
The driver function $f(s, v, y, z, u)$ is a deterministic function which may be written as

$$f(S_t, V_t, Y_t, Z_t, u_t) = (r_t - r) \left( \frac{Z_t^2}{\sqrt{1 - \rho^2}} - Y_t \right) + (\kappa_t - \kappa) \left( \frac{-Z_t^1 \sqrt{V_t}}{\sigma} + \frac{\rho Z_t^2 \sqrt{V_t}}{\sigma \sqrt{1 - \rho^2}} \right)$$

$$+ (\kappa_t \theta_t - \kappa \theta) \left( \frac{Z_t^1}{\sigma \sqrt{V_t}} - \frac{\rho Z_t^2}{\sigma \sqrt{1 - \rho^2} \sqrt{V_t}} \right) - rY_t$$

Thus, $f$ is a linear function of parameter divergence

$$\tilde{u}_t = (r_t - r, \kappa_t - \kappa, \beta_t - \beta), \quad \beta_t \equiv \kappa_t \theta_t$$
Option pricing under parameter uncertainty

- Considering elliptical uncertainty sets
  \[ U = \left\{ u : \tilde{u}^\top \Sigma^{-1} \tilde{u} \leq \chi \right\} \]

- We thus have the quadratic optimisation problems
  \[ H^- = \inf f(\tilde{u}) \quad \text{and} \quad H^+ = \sup f(\tilde{u}) \]
  subject to \( \tilde{u}^\top \Sigma^{-1} \tilde{u} = \chi \)
  with the following solutions

\[
H^\pm(S_t, V_t, Z_t, Y_t) = \pm \sqrt{\chi \, n_t^\top \Sigma^{-1} n_t - rY_t}
\]

\[
\tilde{u}^\pm(S_t, V_t, Z_t, Y_t) = \pm \sqrt{\frac{\chi}{n_t^\top \Sigma^{-1} n_t}} \Sigma n_t
\]

\[
 n_t = \left[ \left( \frac{Z_t^2}{\sqrt{1 - \rho^2} \sqrt{V_t}} - Y_t \right), \left( -\frac{Z_t^2 \sqrt{V_t}}{\sigma} + \frac{\rho Z_t^2 \sqrt{V_t}}{\sigma \sqrt{1 - \rho^2}} \right), \left( \frac{Z_t^1}{\sigma \sqrt{V_t}} - \frac{\rho Z_t^2}{\sigma \sqrt{1 - \rho^2} \sqrt{V_t}} \right) \right]^\top
\]
With the optimal drivers $H^\pm$ we thus have explicit forms for the stochastic differential equations of $D^\pm$ that describe the evolution of the pricing boundaries.

Next, we apply and investigate a number of numerical schemes based on [Bouchard and Touzi, 2004], [Gobet and Lemor, 2008], [Alanko and Avellaneda, 2013] (and modifications thereof) to obtain discrete-time approximations of the solution to the BSDE for $D^\pm$. 
The empirical perspective
The empirical perspective

S&P 500 index

- Underlying asset: 15 years of historical data
- We use daily and weekly variance, $\hat{V}$, as measured with the realised volatility measure from 5-min index observations
The empirical perspective

843 weekly observations from January 3rd, 2000 to February 29th, 2016.
The empirical perspective

- The uncertainty set $U$ is inferred by statistical estimation
- Transition density for $V \sim \text{CIR}$ process is known (non-central chi-squared), however intractable for optimisation
- Use Gaussian likelihood with exact moments – asymptotically normal and efficient estimator [Kessler, 1997]
- $1 - \alpha$ confidence region for $\Theta = (\kappa, \beta, \sigma)$
  \[
  (\Theta - \hat{\Theta})\Sigma^{-1}_{\Theta}(\Theta - \hat{\Theta})^\top \leq \chi^2_3(1 - \alpha)
  \]
  where $\Sigma^{-1}_{\Theta} = I_o$ information matrix from numerical differentiation
- Daily variance: "spiky" time series $\implies$ high estimates of $\kappa \sim 30$ and $\sigma \sim 3$
- Weekly variance: less spikes, more sensible estimates $\kappa \sim 5$ and $\sigma \sim 1$ (comparable to calibration) and lower std. errors
The empirical perspective

S&P 500 call options

- We use bid/offer quotes of S&P 500 call option from a three-year period observed at dates coinciding with the weekly index data ⇒ 157 dates
- Chose a single option being at-the-money at start, and follow this option until maturity (or as long as quotes available) ⇒ four options, ~ 300 quotes
The empirical perspective
The empirical perspective

Conservative pricing bounds

We simulate the optimally controlled value processes (forward with implicit Milstein, backward with explicit scheme based on MARS regression), with $H^+$ for the upper price and $H^-$ for the lower...
The empirical perspective
The empirical perspective

Results

- We find that 98% of the market option prices are within the model-prescribed conservative bounds.

- Bounds fairly symmetrical when option not too far from ATM (III and IV); non-symmetrical when high moneyness (II).

- \[ \Rightarrow \] model unable to capture slope and skew of implies volatilities.
Market implied volatilities of option (II): first date of period in left figure, last date in right figure. Corresponding model-boundaries (dashed lines) and formula-optimal prices (red dotted)
The empirical perspective

Results

For comparison: if we optimise the conventional Heston formula (corresponding to parameter controls restricted to be constants) we cover 40% of the market prices
The empirical perspective
The empirical perspective

Remarks

- After all, we use parameters statistically estimated from the underlying index, not calibrated from option prices.
- Further, use constant set of estimated parameters and uncertainty to predict option prices over the whole three-year period (in practice one would update estimates on regular basis).
- Faced Heston’s model with challenging task: price a dynamical set of market options over long period while taking in information from underlying alone when estimating the model to data.
- In return, allow drift parameters to vary within 95% confidence region as a representation of incompleteness of model, which gives an optimised price range that cover option quotes to some extent.
- When generalising the model, we obtain conservative pricing bounds wide enough to cover most prices, even if some deep in-the-money options still fall outside.
Concluding remarks

- Approach relies on $U$ being a compact set (quadratic form for explicit optimal drivers). Here we infer $U$ from statistical estimation (historical asset prices). Alternatively, one may define $U$ directly as an uncertainty interval based on calibrated option prices. Gaussian noise model $\Rightarrow$ $U$ quadratic form.

- We take the with-spread use of stochastic volatility modes as a starting point, and try to answer how parameter uncertainty can be incorporated and quantified into these models.

- The framework is well suited for multi-asset and multi-factor models (Markovian models in general) and readily adapts to uncertainty in dividend yields [Cohen and Jönsson, 2016].

- Looking forward: hedging at the optimal control functions $\Rightarrow$ super-replication hedging.


