The Fundamental Theorem of Derivative Trading
- Exposition, Extensions, and Experiments

Martin Jönsson
Visiting PhD Candidate, Mathematical Institute
University of Oxford

Department of Mathematical Sciences
Mathematical and Statistical Methods in Insurance and Finance
University of Copenhagen

maj@math.ku.dk
martin.joensson@maths.ox.ac.uk

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Introduction
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- Hedging in real-world practice is, however, a challenging issue: even with a perfect model for the underlying, we still need exact inference for a model-based hedging strategy to be successful.
- Assuming the standard SDE paradigm for the financial market, the issue emanates from the latent nature of volatility.
- We can only hope for a measured quantity: historical volatility from log-returns or implied volatility from observed option prices.
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...and it is hard to believe that the market hysteria which drives the prices of traded options shall capture the market hysteria that drives the prices of the underlying asset.
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The result, as we call The Fundamental Theorem of Derivate Trading, will be the usual suspect for our study:

We derive a more general version, discuss various corollaries and extensions, and test some of its applications in an empirical context.
Basic question: What happens if we use a simple model to delta-hedge some financial derivative?
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Short answer: Not too much if we manage to get the volatility about right. That is the Fundamental Theorem of Derivative Trading.
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About to become a text-book result?

Emanuel Derman
@EmanuelDerman

Nice paper on hedging errors that so few students are taught. All know Black-Scholes, few know how to use it. SSRN papers.ssrn.com/sol3/papers.cf...
Assumptions
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The set-up

- $\mathbb{P}$-dynamics: Objective model for the underlying "real-world" financial market.
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- \( \mathbb{P} \)-dynamics: Objective model for the underlying "real-world" financial market.
- \( \mathcal{M}_i \)-pricing: Option-pricing model assumed to represent collective option market.
- \( \mathcal{M}_h \)-hedging: Hedging model used for our personal purposes.
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- $\mathcal{M}_h$-hedging: Hedging model used for our personal purposes.

We do not assume a connection between the three models: the objective model is fixed, the option model is implied by the option market and the hedging model is our personal choice.
Assumptions

The real-world financial market model

The coefficients $\mu_r(\cdot, \cdot)$ and $\sigma_r(\cdot, \cdot)$ are assumed sufficiently well-behaved for the SDE to have a unique solution.

$\tilde{X}_t$ is defined as the vector $(X_t; \chi_t)$ where $\chi_t$ is a state variable following some dynamics which are irrelevant to what follows.

A money account with price process $B_t$ govern by $dB_t = r_t B_t dt$ where $r_t = r(t, X_t)$ is the locally risk-free rate.
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On a complete, filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), the price processes of \(n\) dividend-paying risky assets \(X_t = (X_{1t}, \ldots, X_{nt})^\top\) following

\[
dX_t = D_X t [\mu_r(t, \tilde{X}_t) dt + \sigma_r(t, \tilde{X}_t) dW_t],
\]

where \(D_X t = \text{diag}(X_{1t}, \ldots, X_{nt})\), and \(W_t\) is a \(n\)-dimensional Brownian motion. The coefficients \(\mu_r(\cdot, \cdot)\) and \(\sigma_r(\cdot, \cdot)\) are assumed sufficiently well-behaved for the SDE to have a unique solution.
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We are completely unaware (or ignorant) of the true form of the real model $\mu_r(\cdot, \cdot), \sigma_r(\cdot, \cdot)$ and the existence of $\chi_t$. Instead, we assume the asset price to follow a local volatility model (under our pricing measure) represented by a deterministic diffusion function in $\mathbb{R}^{n \times n}$

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$\mathcal{M}_h$ corresponds to the belief that a European options have a Markovian pricing rule $V^h_t = V^h(t, X_t)$ where $V^h(t, x)$ is a deterministic $C^{1,2}$-function satisfying Black-Scholes equation

$$r_t V^h = \partial_t V^h + \nabla_x V^h \cdot ((r_t 1 - q_t) \circ x) + \frac{1}{2} \text{tr}[\sigma_h^T D_x \nabla_{xx} V^h D_x \sigma_h],$$

$$V^h(T, x) = g(x),$$
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\( V^h(T, x) = g(x), \)

where \( q_t = q(t, x) \) is the vector of yield rates, \( \cdot \) the dot product, \( \circ \) the Hadamard product, \( \text{tr}[\cdot] \) the trace operator and \( \nabla_x, \nabla^2_{xx} \) the gradient- and Hessian operator.
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We make the same assumptions on the structure of the pricing model that represents the option market: a European option on $X_t$ has a market price process $V^i_t = V^i(t, X_t)$ where $V^i(t, x)$ satisfies Black-Scholes equation with a local volatility function in $\mathbb{R}^{n \times n}$

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$$M_i : \sigma_i(t, x).$$

The local volatility model is consistent with observed market prices: for an observed set of maturity-$T$ strike-$K$ put/call prices, $V^i(t, X_t; T, K)$ matches the market for every $(T, K)$ and state $(t, X_t)$ of the underlying.
Assumptions

**Remark.** The objective model for the financial market \((B_t, X_t)\) is a probabilistic model, while \(M_h\) and \(M_i\) correspond to pricing rules \(V^h(t, x)\), \(V^i(t, x)\) being deterministic functions. In this context, \(M_h\) and \(M_i\) are not probabilistic models per se, although traditionally derived from a probabilistic description of the underlying.
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- We employ $\mathcal{M}_h$ for hedging a European option written on $X_t$, which we buy at time $t = 0$. 
  
We set out to $\Delta$-hedge the option based on our personal hedging model $\mathcal{M}_h$, which yields the fair price $V_h^0$ at time $t = 0$, and furthermore, a hedge position $\Delta_h t \equiv \nabla_x V_h(t, X_t)$ at time $t \in [0, T]$. 

We hold the hedge-portfolio over the entire life-time of the option $[0, T]$. 

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- The asset prices $X_t$ follow the real $\mathbb{P}$-dynamics.
- We employ $\mathcal{M}_h$ for hedging a European option written on $X_t$, which we buy at time $t = 0$.
- The option is acquired on the option market, i.e. it has a value process given by $V_t^i = V^i(t, X_t)$, $t \in [0, T]$ and we pay $V_0^i$ for the option.
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$$P&L_{[0, T]}^h = V_0^h - V_0^i + \frac{1}{2} \int_0^T e^{-\int_0^t r_u du} \text{tr} \left[ D_X \Sigma_{rh}(t, \tilde{X}_t) D_X \nabla^2_{xx} V_t^h \right] dt,$$

where $r_t = r(t, X_t)$ is the risk-free rate, $\nabla^2_{xx}$ is the Hessian operator and

$$\Sigma_{rh}(t, \tilde{X}_t) \equiv \sigma_r(t, \tilde{X}_t)\sigma_r^T(t, \tilde{X}_t) - \sigma_h(t, X_t)\sigma_h^T(t, X_t).$$

is a matrix which takes values in $\mathbb{R}^{n \times n}$. 
Theorem

Proof:

Let \( \Pi_t, t \in [0, T] \) denote the value process of the hedge portfolio long one option with value \( V_i(t) \) and short \( \Delta_h(t) = \partial_x V_h(t, X_t) \) units of \( X_t \), where \( X_t \) follows the real dynamics.

Suppose \( B_t \) is chosen such that the net value is zero due to continuous re-balancing: \( \Pi_h(t) = V_i(t) + B_t - \Delta_h(t) \cdot X_t = 0 \).

From the self-financing condition and previous equation

\[
\frac{d\Pi_h(t)}{dt} = \frac{dV_i(t)}{dt} + r_t B_t - \Delta_h(t) \cdot (dX_t + q_t \circ X_t \, dt) = \frac{dV_i(t)}{dt} - \Delta_h(t) \cdot (dX_t - (r_t - q_t) \circ X_t \, dt) - r_t V_i(t) \, dt.
\]

Next, apply Itô's lemma to \( V_h(t, X_t) \) to obtain

\[
dV_h(t) = \left( \frac{\partial_t V_h(t, X_t)}{2} + \text{tr} \left[ \sigma_r(t, \tilde{X}_t) D\Sigma_X X_t \nabla^2_{xx} V_h(t, X_t) D\Sigma_X X_t \sigma_r(t, \tilde{X}_t) \right] \right) dt + \nabla_x V_h(t, X_t) \cdot dX_t.
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Let \( \Pi_t, t \in [0, T] \), denote the value process of the hedge portfolio \textbf{long} one option with value \( V^i_t \) and \textbf{short} \( \Delta^h_{kt} = \partial_{x_k} V^h(t, X_t) \) units of \( X_{kt} \), \( k = 1, \ldots, n \). That is, \( \Delta^h_t = \nabla_x V^h(t, X_t) \) is the vector of units in the underlying \( X_t \), where \( X_t \) follows the real dynamics.

Suppose \( B_t \) is chosen such that the net value is zero due to continuous re-balancing: \( \Pi^h_t = V^i_t + B_t - \Delta^h_t \cdot X_t = 0. \)

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- Let $\Pi_t, t \in [0, T]$, denote the value process of the hedge portfolio **long one option** with value $V^i_t$ and **short** $\Delta^h_{kt} = \partial_{x_k} V^h(t, X_t)$ **units of** $X_{kt}$, $k = 1, \ldots, n$. That is, $\Delta^h_t = \nabla_x V^h(t, X_t)$ is the vector of units in the underlying $X_t$, where $X_t$ follows the real dynamics.

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From the self-financing condition and previous equation

$$d\Pi_t^h = dV_t^i + r_t B_t dt - \Delta_t^h \cdot (dX_t + q_t \circ X_t dt)$$

$$= dV_t^i - \Delta_t^h \cdot (dX_t - (r_t 1 - q_t) \circ X_t dt) - r_t V_t^i dt. \quad (1)$$
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Proof:

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- From the self-financing condition and previous equation

\begin{equation}
\begin{align*}
  d\Pi^h_t &= dV^i_t + r_t B_t dt - \Delta^h_t \cdot (dX_t + q_t \circ X_t dt) \\
  &= dV^i_t - \Delta^h_t \cdot (dX_t - (r_t 1 - q_t) \circ X_t dt) - r_t V^i_t dt. 
\end{align*}
\end{equation}

- Next, apply Itô’s lemma to $V^h_t = V^h(t, X_t)$ to obtain

\begin{equation}
\begin{align*}
  dV^h_t &= \left( \partial_t V^h_t + \frac{1}{2} \text{tr} \left[ \sigma^T_r(t, \tilde{X}_t) D^T_{X_t} \nabla^2_{xx} V^h_t D_{X_t} \sigma_r(t, \tilde{X}_t) \right] \right) dt + \nabla_x V^h_t \cdot dX_t. 
\end{align*}
\end{equation}
Substitute for $\partial_t V_t^h$ in the Itô differential of $V^h(t, X_t)$ with the Black-Scholes equation for $V^h(t, x)$. This yields the expression

$$0 = -dV_t^h + r_t V_t^h dt + \Delta_t^h \bullet (dX_t - (r_t 1 - q_t) \circ X_t dt)$$

$$+ \frac{1}{2} \text{tr} \left[ D_{X_t} \left( \sigma_r(t, \tilde{X}_t) \sigma_r^\top(t, \tilde{X}_t) - \sigma_h(t, X_t) \sigma_h^\top(t, X_t) \right) D_{X_t} \nabla^2_{xx} V_t^h \right] dt.$$
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$$+ \frac{1}{2} \text{tr} \left[ DX_t \left( \sigma_r(t, \tilde{X}_t)\sigma_r^\top(t, \tilde{X}_t) - \sigma_h(t, X_t)\sigma_h^\top(t, X_t) \right) DX_t \nabla^2_{xx} V^h_t \right] dt.$$

Add to equation (1) for the portfolio dynamics $d\Pi^h_t$, to obtain

$$d\Pi^h_t = dV^i_t - dV^h_t - r_t (V^i_t - V^h_t) dt + \frac{1}{2} \text{tr} [DX_t \Sigma_h(t, \tilde{X}_t) DX_t \nabla^2_{xx} V^h_t] dt$$

$$= e^{\int_0^t r_u du} d \left( e^{-\int_0^t r_u du} (V^i_t - V^h_t) \right) + \frac{1}{2} \text{tr} [DX_t \Sigma_h(t, \tilde{X}_t) DX_t \nabla^2_{xx} V^h_t] dt$$
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$$+ \frac{1}{2} \text{tr} \left[ D_{X_t} \left( \sigma_r(t, \tilde{X}_t)\sigma_r^<(t, \tilde{X}_t) - \sigma_h(t, X_t)\sigma_h^<(t, X_t) \right) D_{X_t} \nabla_{xx}^2 V_t^h \right] dt.$$

Add to equation (1) for the portfolio dynamics $d\Pi_t^h$, to obtain

$$d\Pi_t^h = dV_t^i - dV_t^h - r_t(V_t^i - V_t^h)dt + \frac{1}{2} \text{tr} [D_{X_t} \Sigma_{Rh}(t, \tilde{X}_t) D_{X_t} \nabla_{xx}^2 V_t^h]dt$$

$$= e^{\int_0^t r_udu} d \left( e^{-\int_0^t r_udu} (V_t^i - V_t^h) \right) + \frac{1}{2} \text{tr} [D_{X_t} \Sigma_{Rh}(t, \tilde{X}_t) D_{X_t} \nabla_{xx}^2 V_t^h]dt$$

Discounting back to time zero, integrating the differential over $[0, T]$ and using that $V_T^i = V_T^h = g(X_T)$ give the desired result for the profit-and-loss

$$\text{P&L}^h_{[0, T]} \equiv \int_0^T e^{-\int_0^t r_udu} d\Pi_t^h.$$
Theorem

Remarks

- The proof is a careful application of Itô’s lemma and the Black-Scholes pricing equation. The important point is that our hedging model is assumed to be a local volatility model.

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- The result is a generalization of the work by Karoui, Jeanblanc-Picque & Shreve (1998) (who viewed the result as negative since it proves no bounds for the error), as well as Ahmad & Wilmott (2005).
The proof is a careful application of Itô’s lemma and the Black-Scholes pricing equation. The important point is that our hedging model is assumed to be a local volatility model.

The result is a generalization of the work by Karoui, Jeanblanc-Picque & Shreve (1998) (who viewed the result as negative since it proves no bounds for the error), as well as Ahmad & Wilmott (2005).

The moniker *The Fundamental Theorem of Derivative Trading* was coined (we think) by Andreasen (2003).
The Theorem

Remarks

The theorem can be extended in various directions: e.g. if
\[ V_t = V(t, X_t, A_t) \]
is the price of an Asian option on the continuous average \( A_t \) and underlying \( X_t \), the theorem remains form-invariant.
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- Clearly, the profit-and-loss changes sign if we consider the situation where we go short the option, long the underlying.
Corollaries
Corollaries

▶ The Real Hedge:

If we perfectly match the real dynamics in our ∆-hedge;

\[ \sigma_h(t, X_t) = \sigma_r(t, \tilde{X}_t) \text{ a.s. for all } t \in [0, T], \]

then our terminal profit-and-loss is deterministic

\[ P&L_h = r[0, T] = V_{r0} - V_{i0} \]

provided we hold the hedge portfolio until maturity (and hedge continuously in time).

▶ However, day-to-day fluctuations of the hedge portfolio still evolve erratically:

\[ d\Pi_h = \frac{1}{2} tr\left[ D^2 X_t \Sigma ri(t, \tilde{X}_t) D^2 X_t \nabla^2_{xx} V_i t \right] dt + \nabla_x (V_i t - V_r t)_\cdot \left\{ (\mu_r t - r t 1 + q t) \cdot X_t dt + D^2 X_t \sigma_r(t, \tilde{X}_t) dW_t \right\}. \]

That is, the mark-to-market P&L dynamics is driven by a \( dW_t \)-term (the effect of which is clearly visible when hedging with discrete, daily rebalancing).
Corollaries

- **The Real Hedge:** If we perfectly match the real dynamics in our Δ-hedge; $\sigma_h(t, X_t) = \sigma_r(t, \tilde{X}_t)$ a.s. for all $t \in [0, T]$, then our terminal profit-and-loss is deterministic

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\[
d\Pi_t^{h=r} = \frac{1}{2} \text{tr}[D_{X_t} \Sigma_{ri}(t, \tilde{X}_t) D_{X_t} \nabla_{xx}^2 V_t^i] dt
\]

\[
+ \nabla_x (V_t^i - V_t^r) \cdot \left\{ (\mu_t^r - r_t 1 + q_t) \circ X_t dt + D_{X_t} \sigma_r(t, \tilde{X}_t) dW_t \right\}.
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$$+ \nabla_x (V_t^i - V_t^r) \cdot \left\{ (\mu^r_t - r_t \mathbf{1} + q_t) \circ X_t dt + D_x X_t \sigma_r(t, \tilde{X}_t) dW_t \right\}.$$ 

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▶ The Implied Hedge:

Suppose instead that we match the implied volatility; \( \sigma_h(t, X_t) = \sigma_i(t, X_t) \) \( \text{a.s.} \) for all \( t \in [0, T] \), we then obtain

\[
P&L_h = \frac{1}{2} \int_0^T e^{-\int_0^t r_u \, du} tr\left[ D_X X_t \Sigma ri(t, \tilde{X}_t) D_X X_t \nabla^2 xV_i(t) \right] dt.
\]

This is stochastic although the portfolio value process is of finite variation: we do not have a stochastic integral in \( \Pi_h = i_t \) w.r. to the Brownian motion.

When hedging at implied volatility we don't know exactly which terminal P&L we will get. But we will have bleeding due to our false model, not blow-ups!
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$$P&L^{h=i}_{[0, T]} = \frac{1}{2} \int_0^T e^{-\int_0^t r_udu} \text{tr}[DX_t \Sigma_{ri}(t, \tilde{X}_t)DX_t \nabla^2_{xx} V^i_t] dt.$$
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As for profitability

- The real hedge gives $P&L = V_0' - V_0'$ (provided continuous hedging): we simply go short the hedge portfolio if $V_0' < V_0'$ to make a certain profit.
Corollaries

As for profitability

- The real hedge gives $P&L = V_0^r - V_0^i$ (provided continuous hedging): we simply go short the hedge portfolio if $V_0^r < V_0^i$ to make a certain profit.

- The implied hedge: $P&L = \frac{1}{2} \int_0^T e^{-\int_0^t r_u du} \text{tr}[D_X(t) \Sigma_{ri}(t, \tilde{X}_t) D_X(t) \nabla_{xx} V_t^i] dt$. By property of the trace operator, we may write

$$\text{tr}[D_X(t) \Sigma_{ri}(t, \tilde{X}_t) D_X(t) \nabla_{xx} V_t^i] = X_t^\top (\Sigma_{ri}(t, \tilde{X}_t) \circ \nabla_{xx} V_t^i) X_t$$
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\]

and we see that the P&L is positive (a.s.) if $\Sigma_{ri}(t, \tilde{X}_t) \circ \nabla_{xx}^2 V_t^i$ is a positive definite matrix at all times (a.s.). By the Schur product theorem, this is the case if $\Sigma_{ri}(t, \tilde{X}_t)$ and $\nabla_{xx}^2 V_t^i$ are both positive definite matrices individually.
Corollaries

- Hedging with real volatility; a deterministic terminal P&L but we get there in an erratic way, or hedging with implied volatility; a stochastic terminal P&L, but we get there in a smooth way.
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- Hedging with real volatility; a deterministic terminal P&L but we get there in an erratic way, or hedging with implied volatility; a stochastic terminal P&L, but we get there in a smooth way.

- For a demonstration of this point we run a simulation experiment based on Wilmott and Ahmad (2005).
Corollaries

**Wilmott’s Hedge Experiment:** In a standard Black-Scholes world with a single stock, we Δ-hedge a European call option with either $\sigma_h = \sigma_r$ or $\sigma_h = \sigma_i$. Ten simulated paths of the stock give P&L realisations:
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Left: Hedging with real volatility. Right: Hedging with implied volatility.

The points of previous slides are clear: erratic P&L paths of \( \mathcal{O}(dW) \) with deterministic terminal value (almost; due to discrete rebalancing) or smooth, positive paths of \( \mathcal{O}(dt) \) with implied volatility (since BS-gamma \( \partial_{xx} V_t^i > 0 \)).
Which Free Lunch Would You Like Today, Sir?

- Both strategies of the experiment simply suggest that if we know the real volatility, e.g. from historical estimation $\sigma_{\text{hist}} \approx \sigma_r$, and its relation to the implied; $\sigma_i < \sigma_{\text{hist}}$ ($\sigma_i > \sigma_{\text{hist}}$), then we can make a certain profit if we go long (short) the hedge portfolio.
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▶ So, what happens if we run Wilmott’s experiment on market data; are there any empirical support for these results?
Experiments
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We set out to investigate the performance of the Δ-hedge in an empirical setting, based on (I) forecasted implied volatilities and (II) forecasted actual volatilities.
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For forecasts of daily implied volatility $\sigma^{imp}_t$ over the portfolio lifetime, we use the ATM implied volatility of corresponding tenor at the initiation date of the hedge.
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For forecasts of daily implied volatility $\sigma_{t}^{imp}$ over the portfolio lifetime, we use the ATM implied volatility of corresponding tenor at the initiation date of the hedge.

For the real volatility $\sigma_{t}^{act}$ we use forecasts of an EGARCH(1,1) model fitted to the S&P500 index from the previous portfolio period.
Experiments
We investigate 36 hedge portfolios of three-month call options on S&P500, purchased at-the-money and kept until expiry.
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- We investigate 36 hedge portfolios of three-month call options on S&P500, purchased at-the-money and kept until expiry.

- With daily rebalancing, we set up self-financing hedge portfolios based on the Black-Scholes delta, long or short the call option depending on the sign of \((\sigma_t^{act} - \sigma_t^{imp})\), and we hold the portfolios until maturity.
Experiments

Left: Hedging with EGARCH(1,1) volatility. Right: Hedging with implied volatility.
Solid lines show positions where we initially go short, dotted where we initially go long.

The paths from the implied hedge appears to be smoother, but there is no certain positive profit to be made for neither of the two strategies.
Experiments

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Mean ($m$)</th>
<th>Std. Dev. ($sd$)</th>
<th>Hypothesis Tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hedge error, actual volatility</td>
<td>7.7</td>
<td>17.3</td>
<td>Q: $m = 0$?</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>A: No; $p$-value = 1%</td>
</tr>
<tr>
<td>Hedge error, implied volatility</td>
<td>7.7</td>
<td>16.6</td>
<td>Q: $m = 0$?</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>A: No; $p$-value = 1%</td>
</tr>
<tr>
<td>Quadratic variation, actual volatility</td>
<td>1.2</td>
<td>2.1</td>
<td>Q: $sd_{act} = sd_{imp}$?</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>A: Yes; $p$-value = 55%</td>
</tr>
<tr>
<td>Quadratic variation, implied volatility</td>
<td>0.81</td>
<td>2.0</td>
<td>Q: $m_{QV_{act}} = m_{QV_{imp}}$?</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>A: No; $p$-val. = 1.4%</td>
</tr>
</tbody>
</table>

**Table:** Summary statistics and hypothesis tests for different hedge strategies.
Experiments

- Although no certain positive terminal P&L for every path, we have a significant positive payout on the average.
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- If we are concerned about the terminal hedge error only: the strategies yield equal mean and standard deviation, i.e. the strategies are not significantly different in riskiness.

\[ \sigma_h = \sigma_r, \quad \text{which gives} \quad P&L = V_r - V_i. \] Neither can we hope for getting the sign right of \((\sigma_r - \sigma_i)\), which would guarantee P&L paths positive at termination.
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- But when hedging with implied volatility, the quadratic variation of the P&L is more than halved compared to using actual volatility. This should make you sleep better at night.

- At least we got one thing right: we may reduce the erratic behaviour of our daily P&L-paths by hedging with the implied volatility. However, we can not hope for a bang-on estimate of the real volatility; $\sigma_h = \sigma_r$, which would give $P&L = V_0^r - V_0^i$. Neither can we hope for getting the sign right of $(\sigma_t^r - \sigma_t^i)$, which would guarantee P&L paths positive at termination.
Extensions: The Case of Jumps
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A natural extension of our previous setting with continuous processes is to allow for jumps in the asset price dynamics. As an example, let the underlying assets $X_t = (X_{1t}, \ldots, X_{nt})$ have real dynamics given by the jump-diffusion

$$dX_t = DX_{t-} [\mu_r(t, \tilde{X}_t) dt + \sigma_r(t, \tilde{X}_t) dW_t + \int_y \tilde{N}(dy, dt)]$$

where $\tilde{N}(dy, dt)$ is a compensated Poisson random measure, independent of $W_t$ and with compensator $\nu(dy)dt$. 
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\]

where \( \tilde{N}(dy, dt) \) is a compensated Poisson random measure, independent of \( W_t \) and with compensator \( \nu(dy)dt \).

\( \tilde{N}(dy, dt) \) governs the (relative) jumps of \( X_t \):

\[
\Delta X_t = DX_{t-} \int_y y\tilde{N}(dy, \{t\}) = (X_{t-} \circ y_t)I_{\{\Delta X_t \neq 0\}}
\]

where \( y_t \) is the value such that \( N(\{y_t, t\}) = 1 \) if exists, arbitrary otherwise.
Extensions: The Case of Jumps

Under the pricing and hedging models

\[ \mathcal{M}_h : \{ \sigma_h(t, x), \nu_h(dy) \} , \]
\[ \mathcal{M}_i : \{ \sigma_i(t, x), \nu_i(dy) \} , \]

we assume European options to permit Markovian pricing rules:
\[ V^h = V^h(t, X_t) \] for deterministic \( C^{1,2} \)-function \( V^h(t, x) \) that satisfies the pricing PIDE

\[
\begin{align*}
  r_t V^h &= \partial_t V^h + \nabla_x V^h \cdot ((r_t 1 - q_t) \circ x) + \frac{1}{2} \text{tr} [\sigma^T_h D_x \nabla^2_x V^h D_x \sigma_h] \\
  &\quad + \int_y \left( V^h(t, x + x \circ y) - V^h(t, x) - (x \circ y) \cdot \nabla_x V^h \right) \nu_h(dy),
\end{align*}
\]

while \( V^i(t, x) \) satisfies the corresponding equation under \( \mathcal{M}_i \).
Theorem

The Fundamental Theorem of Derivative Trading for Jumps. Assume we acquire at \( t = 0 \) a European option with payoff \( g(X_T) \) for the market price \( V^i_0 \). Suppose we \( \Delta \)-hedge our position based on the model \( M_h \), leading to the fair price \( V^h_0 \). Then, the present value of the P&L incurred over the option's entire lifetime \([0, T]\) is

\[
P&L^h_{[0, T]} = V^h_0 - V^i_0 + \frac{1}{2} \int_0^T e^{-\int_0^t r_u \, du} \text{tr} \left[ D_X t \Sigma_{rh}(t, \tilde{X}_t) D_X t \nabla^2_{xx} V^h_t \right] dt + P&L^J_{[0, T]}
\]

where \( P&L^J_{[0, T]} \) is the profit-&-loss contribution that originates from the jumps of the real dynamics (equation 2) and hedging model (equation 3)

\[
P&L^J = \sum_{0 \leq t \leq T} \Delta X_t \neq 0 e^{-\int_0^t r_u \, du} \left( V^h(t, X_t^{-} + \Delta X_t) - V^h(t, X_t^{-}) - \Delta X_t \cdot \nabla_x V^h \right) (2)
\]

\[- \int_{y,t} e^{-\int_0^t r_u \, du} \left( V^h(t, X_t^{-} \circ (1 + y)) - V^h(t, X_t^{-}) - (X_t^{-} \circ y) \cdot \nabla_x V^h \right) \nu_h(dy) dt. \] (3)
Extensions: The Case of Jumps

Remarks

▶ The proof: in a similar fashion as for the case without jumps but with the pricing PIDE and Itô’s lemma for a $n$-dimensional semimartingale.
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- In fact, we may relax our assumptions to allow for the objective $X_t$ to be a general semimartingale and dropping the assumption on $M_i$ as the market price process $\{V^i_t\}_{t\in[0,T]}$ enters only through the initial value $V^i_0$, provided we look at the case when hedging throughout $[0, T]$. 

- The terminal hedge error decomposes into three parts originating from (I) the option’s model-to-market price difference, (II) the continuous quadratic variation of $X_t$ from the real- and hedging model, (III) the jumps $\Delta X_t$ from the real dynamics and the hedging model jump-distribution.
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Extensions: The Case of Jumps

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- Similarly as in the continuous setting, we may hedge matching the real dynamics:

\[
P\&L_{[0,T]}^{h=r} = V_0^r - V_0^i + P\&L_{[0,T]}^J
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no longer deterministic, but with expected value

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- Matching the option market:

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no longer with smooth P&L paths of zero quadratic variation; the portfolio value will have jumps

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\[ \Delta \Pi_t^{h=i} = \Delta V_t^i - \Delta X_t \cdot \nabla_x V_t^i. \]

- One may argue that these points are in better agreement with our empirical results.
Extensions: The Case of Jumps

Remarks

Finally, if we consider the case where the pricing function $V^h(t, x)$ is convex in $x$ (e.g. plain vanilla puts and calls), we have that

$$\sum_{\Delta x_t \neq 0} e^{-\int_0^t r_u \, du} \left( V^h(t, x_{t-} + \Delta x_t) - V^h(t, x_{t-}) - \Delta x_t \cdot \nabla x V^h \right)$$

yields a positive contribution to the long-position P&L every time $X_t$ jumps (in either direction). Conversely, for a short position, our hedge portfolio takes a "hit" every time a jump occurs: in Talebian terms, short selling convex payoff options corresponds to...
Extensions: The Case of Jumps

... “picking up pennies in front of a steam roller”.
Conclusion
We generalize the fundamental theorem of derivative trading, a result that quantifies the profit-&-loss of a $\Delta$-hedge under a misspecified volatility.
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▶ We show that hedging with implied volatility yields smooth P&L paths with changes of $O(dt)$ while any other hedging volatility yields erratic paths of $O(dW)$ (in the continuous setting: add jumps otherwise).

▶ We find empirical support for our theoretical results: there is evidence that hedging at implied volatility does yield smoother P&L paths.

▶ However, the most conspicuous implication of the theorem – the ease with which arbitrage can be made if the relative sizes of $\sigma_{\text{hist}}$, $\sigma_{\text{imp}}$ are known – is not as convincing: even if we find a positive average, the P&L may readily turn negative.

▶ Finally, we extend the continuous models to affiliate jumps and may argue that this setting better explains the results of our empirical investigation.
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Thank you for your attention!

For complete list of references, please see draft of our paper available at SSRN: