Abstract—This paper addresses the problem of decentralised simultaneous localisation and map building for a team of agents where the communication bandwidth is limited. We present an extension to current approaches that enables multiple vehicles to acquire a joint map, but which can cope with communication bandwidth limitations. Nettleton’s approach uses a hybrid information filter/Covariance Intersection algorithm on each communication link to manage the inter-vehicle communication and ensure that information vehicles share does not get ‘double counted’. The Covariance Intersection algorithm is a highly conservative method for managing double counting and its use can produce highly uncertain maps. We introduce a novel and more efficient tool, called Bounded Covariance Inflation, for managing the double counting (or rumour propagation) problem. We show that the parameters required by the new approach can be determined locally by each vehicle and therefore the decentralised nature of the network is not compromised. We provide experimental results that illustrate the effectiveness of our approach in comparison with the original approach of Nettleton et al.

I. INTRODUCTION

Simultaneous localisation and map building (SLAM) is the problem of acquiring a map of an environment whilst simultaneously localising relative to the map. Until recently all approaches to solving the SLAM problem had used a single vehicle to explore the environment (e.g. [7]). However, Nettleton et. al [6] have proposed a decentralised multi-vehicle approach in which a team of vehicles acquires a joint map. This approach to map building is commonly referred to as decentralised SLAM (DSLAM). The approach is described in detail in [6] and is only briefly summarised here.

Within the DSLAM approach the map is represented by an estimate and covariance of this estimate. Each vehicle maintains an estimate of its own state (i.e. position and orientation) along with location estimates of mapped features. These estimates are generated by a Extended Kalman filter (called the local SLAM filter) using observed, predicted and communicated information. The augmented state vector for the vehicle and map at time $t_i$ given observations up to time $t_j$ is written as $\hat{x}(i \mid j)$ with an associated covariance $P(i \mid j)$. However, when written in its equivalent information form, the system is described by the information vector $\hat{y}(i \mid j)$ and information matrix $Y(i \mid j)$:

$$\hat{y}(i \mid j) \triangleq P^{-1}(i \mid j)\hat{x}(i \mid j) ,$$
$$Y(i \mid j) \triangleq P^{-1}(i \mid j) .$$

Communicating entire maps may not be possible when the communication bandwidth is limited and the constant time communication (CTC) algorithm, proposed in [6], is an alternative approach which incrementally communicates sub-maps between vehicles. Within the CTC approach, sub-maps are represented in information form, and subsets of the information is communicated selectively along channels which are point-to-point links with no loops in the network. The channel filter is used to manage communication of information between vehicles. Its main purpose is to keep track of information previously communicated on the channel to ensure data is not double counted. When a vehicle communicates a sub-map the channel filter is used to subtract the common information and pass only the increment of new information to the receiving vehicle’s SLAM filter.

Nettleton’s approach uses a hybrid information filter/Covariance Intersection algorithm on each communication link to manage the inter-vehicle communication. However, Covariance Intersection (CI) [5] is a highly conservative method for managing double counting and its use can produce highly uncertain maps. An alternative approach, which is introduced in this paper, uses a novel tool, called Bounded Covariance Inflation (BCInf) in place of CI. Bounded Covariance Inflation uses knowledge of the bounds on estimate correlations between a vehicle and its channel to fuse sub-maps more efficiently than CI. The parameters required by the new approach can be determined locally by each vehicle and therefore the decentralised nature of the network is not compromised. Further, our approach places hardly any communication burden on the data links, a crucial characteristic for limited bandwidth communications.

Section II introduces Bounded Covariance Inflation (BCInf), a mechanism for creating conservative covariance matrices for random variables for which upper and/or lower bounds on the cross correlations are known. Section III describes how bounds on the covariances of maps held at different vehicles can be determined efficiently and locally without the need for inter-agent modelling. Section IV details the BCInf steps in the DSLAM constant time communication algorithm. Finally, in Section V we provide experimental results that illustrate the effectiveness of our approach in comparison with the original approach of Nettleton et al..
II. BOUNDED COVARIANCE INFLATION

This section introduces the general theory of Bounded Covariance Inflation (BCInf), a mechanism for creating conservative covariance matrices from bounded cross correlations. When \( \hat{u} \) is an estimate for the state \( u \) then \( P_{uu}^* \) is a conservative matrix for the covariance of \( \hat{u} - u \) if:

\[
P_{uu}^* \geq E[\bar{\hat{x}}\bar{\hat{x}}^T] \quad \text{where} \quad \bar{\hat{x}} = \bar{x} - x.
\]

The symbol \( \geq \) denotes positive semi-definite. When \( u \) is composed by stacking two vectors, \( x \) and \( y \) say, with corresponding covariances \( P_{xx} \) and \( P_{yy} \) respectively, covariance inflation is the procedure by which \( P_{uu}^* \) can determined from \( P_{xx} \) and \( P_{yy} \) when the correlation between \( x \) and \( y \) is unknown but bounded.

Readers familiar with Covariance Intersection (CI) [5] may find the following paragraph a useful introduction to covariance inflation. Consider a state space vector \( u \) segmented into vectors \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^n \). Vehicle A has an estimate for \( x \) but knows nothing about \( y \) and vehicle B has an estimate for \( y \) and knows nothing about \( x \). Initially vehicle A’s information vector and matrix are:

\[
\hat{y}_A = \begin{bmatrix} P_{xx}^{-1}\bar{x} \\ 0 \end{bmatrix}, \quad Y_A = \begin{bmatrix} P_{xx}^{-1} & 0 \\ 0 & 0 \end{bmatrix}.
\]

and vehicle B’s are:

\[
\hat{y}_B = \begin{bmatrix} 0 \\ P_{yy}^{-1}\bar{y} \end{bmatrix}, \quad Y_B = \begin{bmatrix} 0 & 0 \\ 0 & P_{yy}^{-1} \end{bmatrix}.
\]

Vehicle A communicates its information vector \( \hat{y}_A \) to vehicle B and vehicle B fuses both potentially correlated estimates, \( \hat{y}_A \) and \( \hat{y}_B \), using CI. The fused covariance \( P \) is, using CI:

\[
P_{uu}^* = [(1-\omega)Y_A + \omega Y_B]^{-1} = \begin{bmatrix} P_{xx} & 0 \\ 0 & P_{yy} \end{bmatrix} \begin{bmatrix} P_{xx}^{-1} & 0 \\ 0 & P_{yy}^{-1} \end{bmatrix}.
\]

\( P_{uu}^* \) is a conservative covariance matrix for the joint estimate \( \hat{u} \) over \( x \) and \( y \).

The parameter \( \omega \) can be chosen to minimise the overall uncertainty encoded by \( P^* \). The determinant \( \det[P^*] \) is a popular measure of uncertainty and can be shown to be a function of the dimensions of \( P_{xx} \) and \( P_{yy} \) only and \( \omega = \omega_* \), where \( \omega_* = \frac{n}{m+n} \) minimises \( \det[P^*] \). Note that, when \( m = n \) we get the familiar “double the diagonal” rule for guaranteeing that a covariance matrix is conservative.

What if there is knowledge of the bounds on the cross correlation between \( x \) and \( y \)? Their cross covariance \( P_{xy} \) is not explicitly known, but \( P_{xy} \) is bounded:

\[
[P_{xy} - D_{xy}]^T P_{xx}^{-1} [P_{xy} - D_{xy}] \leq S^2 P_{yy}
\]

or, equivalently:

\[
\forall \bar{x}, \bar{y} \in \mathbb{R}^m, \quad [\bar{x}^T R_{xx}^T (P_{xy} - D_{xy}) R_{yy} \bar{y}] \leq S
\]

for unit vectors \( \bar{x} \) and \( \bar{y} \), some “centred” matrix \( D_{xy} \), “matrix spread” \( S \) and sphering matrices \( R_{xx} \) and \( R_{yy} \) such that \( P_{xx}^{-1} = R_{xx}^T R_{xx} \) and \( P_{yy}^{-1} = R_{yy}^T R_{yy} \). When \( D_{xy} = 0 \) then

\[
S \quad \text{is the correlation coefficient. In general, we choose } D_{xy} \text{ so that } S \text{ is as small as possible (see Figure 1).}
\]

Then it is possible to find a conservative covariance \( P_{BCInf}^* \) for all possible joint covariances \( P \) defined by:

\[
P = \begin{bmatrix} P_{xx} & P_{xy} \\ P_{xy}^T & P_{yy} \end{bmatrix}.
\]

Define \( P_{BCInf}^* \): \(^1\)

\[
P_{BCInf}^* \triangleq \begin{bmatrix} (1+KS)P_{xx} & D_{xy} \\ D_{xy}^T & (1+\frac{S}{K})P_{yy} \end{bmatrix}
\]

The positive value \( K \) is called the inflation factor and is chosen to minimise the overall uncertainty encoded by the covariance matrix and has an analogous role to \( \omega \) in CI.

Theorem 1: \( P_{BCInf}^* \geq P \)

Proof: For any \( K > 0 \),

\[
\begin{bmatrix} KP_{xx} - P_{xy} & -P_{xy} \\ -P_{xy}^T & \frac{1}{K}P_{yy} \end{bmatrix} \geq 0 \iff \begin{bmatrix} P_{xx} & P_{xy} \\ P_{xy}^T & P_{yy} \end{bmatrix} \geq 0.
\]

Therefore, for all \( P_{xy} \) satisfying Eqn (2):

\[
P_{BCInf}^* - P \geq 0 \iff \begin{bmatrix} KSP_{xx} & SP_{xy} \\ SP_{xy} & D_{xy} - P_{yy} \end{bmatrix} \geq 0
\]

\[
\iff \begin{bmatrix} P_{xx} & P_{xy} \\ P_{xy}^T & P_{yy} \end{bmatrix} \geq 0.
\]

Let \( V \) be the arbitrary stacked vector \( V^T = [R_{xx} \omega R_{yy}] \).

\(^1\)It has been recently been brought to our attention that Hanebeck et al. [3] developed a similar expression to Eqn (4). However, their paper only considers upper bounds on the absolute cross correlations (i.e. cases when \( D_{xy} = 0 \)).
Using Eqn (2) we get:

\[
V^T \begin{bmatrix}
SP_{xx} & P_{xy} - D_{xy} \\
D_{xy} & SP_{yy}
\end{bmatrix} V
= \begin{bmatrix}
S(x^2 + y^2) + 2|x||y|\alpha & \rho \sqrt{\max\text{eig}}(R_{xx}C_{xx}P_{aa}^{-1}C_{aa}R_{xx}) \\
\rho \sqrt{\max\text{eig}}(R_{yy}C_{yy}P_{aa}^{-1}C_{yy}R_{yy}) & \max\text{eig}(R_{xx}C_{xx}P_{aa}^{-1}C_{aa}R_{xx})
\end{bmatrix}
\]

where \( \rho \) is the cross correlation coefficient. Comparing the above inequality with Eqn (2) we observe that the right hand side is a known bound for the cross correlation bound \( S \) in Eqn (2) first rewrite Eqn (7):

\[
P_{xy} = C_{xx}\sqrt{P_{aa}^{-1}} \left[ C_{yy}\sqrt{P_{aa}^{-1}} \right]^T
\]

and use the Cauchy-Schwartz inequality:

\[
|x^TYR_{xx}P_{xy}R_{yy}| \leq \sqrt{\max\text{eig}(R_{xx}C_{xx}P_{aa}^{-1}C_{aa}R_{xx})} \times \sqrt{\max\text{eig}(R_{yy}C_{yy}P_{aa}^{-1}C_{yy}R_{yy})}
\]

III. COUPLING SCALARS

How do we determine upper and lower bounds on the cross correlations? Two correlated estimates \( \hat{x} \) and \( \hat{y} \) for state vectors \( x \) and \( y \) respectively can be decomposed into orthogonal random vectors \( \alpha, \beta_x \) and \( \beta_y \) by Gram-Schmidt orthogonalisation [1]:

\[
\begin{align*}
\hat{x} &= C_{xx}P_{aa}^{-1}\alpha + \beta_x \\
\hat{y} &= C_{yy}P_{aa}^{-1}\alpha + \beta_y
\end{align*}
\]

where \( C_{xx} \) and \( C_{yy} \) are the cross covariances between \( x \) and \( \alpha \) and between \( y \) and \( \alpha \) respectively. Since \( \beta_x \) and \( \beta_y \) are orthogonal then:

\[
P_{xy} = C_{xx}P_{aa}^{-1}C_{yy}.
\]

So, the cross covariance between two random vectors is the information shared between the two vectors projected onto the vector spaces.

To obtain an expression for the minimum cross correlation bound \( S \) in Eqn (2) first rewrite Eqn (7):

\[
P_{xy} = \left[ C_{xx}\sqrt{P_{aa}^{-1}} \right] \left[ C_{yy}\sqrt{P_{aa}^{-1}} \right]^T
\]

and use the Cauchy-Schwartz inequality:

\[
|x^TYR_{xx}P_{xy}R_{yy}| \leq \sqrt{\max\text{eig}(R_{xx}C_{xx}P_{aa}^{-1}C_{aa}R_{xx})} \times \sqrt{\max\text{eig}(R_{yy}C_{yy}P_{aa}^{-1}C_{yy}R_{yy})}
\]

Comparing the above inequality with Eqn (2) we observe that the right hand side is a known bound for the cross correlation bound \( S \). The scalar:

\[
\Upsilon = \sqrt{\max\text{eig}(R_{xx}C_{xx}P_{aa}^{-1}C_{aa}R_{xx})}
\]

is called the coupling scalar for \( x \) and is independent of \( y \). The coupling scalars can be calculated locally. Thus, when a vehicle C receives two messages, one from each of vehicles A and B, comprising the information vector, matrix and coupling scalar then a bound on the correlation between the estimates in these messages is (see Figure 2):

\[
S = \Upsilon_{AC} \times \Upsilon_{BC}.
\]

The key point of this section is that the cross correlation between two random vectors can be bounded by the product of just two scalars. This is crucial for limited bandwidth communication applications for which book keeping messages such as the coupling scalar must be kept to a minimum. A limited bandwidth application is investigated in Section IV.

As a final note, the coupling scalar can be interpreted as the fraction of the covariance matrix which is the correlated part \( C_{xx}P_{aa}^{-1}\alpha \) of \( \hat{x} \). From Eqn (6) we see that it would be possible to communicate the correlated part and the uncorrelated part \( \beta_x \) of the estimate \( \hat{x} \) separately. The receiving vehicle would then be able to fuse \( \hat{x} \) into its own estimate more efficiently than the method described above as only the correlated part of \( \hat{x} \) would have to be inflated prior to fusion [4]. However, this alternative approach would involve nearly twice the communication load compared to the coupling scalar approach.
and this is unacceptable for limited bandwidth communication applications.

IV. DSLAM APPLICATION

This section describes an application of bounded covariance inflation to the multi-vehicle DSLAM problem. The various Kalman filter update steps are described much as in [6] but for a constant time communication strategy using BCInf in place of CI. The reader is invited to compare the appropriate section in [6] in order to appreciate more fully the differences and similarities in these approaches.

We will consider the communication of sub-maps from vehicle A to vehicle B via channel C. Prior to communication at time \( t_k \), the vehicles’ local filters and channel will contain their own information vectors and matrices \( \{ \hat{y}_A(k | k), Y_A(k | k) \} \), \( \{ \hat{y}_B(k | k - 1), Y_B(k | k - 1) \} \) and channel \( \{ \hat{y}_C(k | k - 1), Y_C(k | k - 1) \} \).

A sub-map of some arbitrary size \( M^2 \), where \( 0 \leq M \leq N \), is formed by agent A using the \( M \) features with the greatest information gain [6]. In order to do this, it is necessary to firstly define a matrix \( G_m \) that selects from the map a sub-map of dimension \( M^2 \). Suppose that vehicle A chooses to communicate the sub-vector \( G_m \hat{x}_A(k | k) \) where \( G_m \) is an \( M \times N \) selection matrix. The corresponding covariance sub-matrix is \( G_m P_A(k | k) G_m^T \).

When the channel filter receives the information vector and matrix:

\[
\hat{y}_A(k | k) = Y_A(k | k)G_m \hat{x}_A(k | k),
\]

\[
Y_A(k | k) = [G_m P_A(k | k) G_m^T]^{-1},
\]

from the local filter it incorporates this information into the SLAM estimate. In order to do this, it is necessary to convert the sub-map to the dimension of the entire SLAM state by padding elements with zeros. We will assume that the channel and local SLAM mapped features are similarly ordered so that we may use the inverse transformation of \( G_m \) as in [6]. The channel filter fuses the estimates using BCInf:

\[
\begin{align*}
\hat{y}_C(k | k) &= \frac{\hat{y}_C(k | k - 1)}{1 + KS} + \frac{G_m^T \hat{y}_A(k | k)}{1 + \frac{K}{R}} , \\
Y_C(k | k) &= \frac{Y_C(k | k - 1)}{1 + KS} + \frac{G_m^T Y_A(k | k) G_m}{1 + \frac{K}{R}} .
\end{align*}
\]

The value for \( K \) is chosen to minimise the uncertainty encoded by the determinant of \( Y_C(k | k) \) and is found numerically (using the Matlab function \( \text{fminbnd} \), for example).

How do we calculate the correlation coefficient \( S \)? In the Appendix we show that the covariance between the newly communicated sub-map estimate and the channel estimate obeys:

\[
\text{Cov}[G_m^T \hat{x}_A(k | k), \hat{x}_C(k | k - 1)] = G_m^T G_m Y_A^{-1}(k | k) Y_C(k | k - 1) Y_C(k | k - 1)^{-1} .
\]

Setting \( \alpha = \hat{x}_C \) in Eqn (6) and then comparing the above equation with Eqn (7), we get:

\[
\begin{align*}
C_{AC} &= G_m^T G_m Y_A(k | k)^{-1} , \\
C_{CC} &= Y_C(k | k - 1)^{-1} .
\end{align*}
\]

The coupling scalar for the channel filter is 1 and therefore, using the results of Section III, the cross correlation bound is the coupling scalar for the local vehicle:

\[
S = \sqrt{\text{max}\text{eig}[R_{AA}^T C_{AC} Y_C C_{CA} R_{AA}]}]
\]

where:

\[
R_{AA}^{-1} = \sqrt{G_m^T Y_A(k | k)G_m} .
\]

When vehicle B receives the new sub-map information it updates its own channel filter using exactly the same update steps as above. Once updated, it calculates the increment of new information it has just received from vehicle A that has not already been fused locally at vehicle B:

\[
\begin{align*}
Y_{AB}(k | k) &= Y_C(k | k) - Y_C(k | k - 1) , \\
\hat{y}_{AB}(k | k) &= \hat{y}_C(k | k) - \hat{y}_C(k | k - 1) .
\end{align*}
\]

This information increment is then sent to the local filter to be fused into the SLAM estimate.

When the local filter receives the sub-map information \( Y_{AB}(k | k) \) and \( \hat{y}_{AB}(k | k) \) from the channel filter it uses this information in the SLAM estimate. In order to do this, it is necessary to firstly define a matrix \( G_m \) that augments the map to the dimension of the entire SLAM state by padding elements with zeros. The update is then done by adding the new information from the channel:

\[
\begin{align*}
\hat{y}(k | k) &= \hat{y}(k | k - 1) + G_s \hat{y}_{AB}(k | k) , \\
Y(k | k) &= Y(k | k - 1) + G_s Y_{AB}(k | k) G_s^T .
\end{align*}
\]

The SLAM update steps described by Eqns (11) to (14) are identical to the update steps in Nettleton’s approach. These equations were justified through intuition in [6]. A mathematical proof for them is presented in the Appendix.
V. SIMULATION RESULTS

The BCInf constant time communication algorithm was implemented and run in a two vehicle simulation to test and evaluate its performance with respect to a number of other communication strategies. The different communication strategies used were:

1) Transmission of the complete information map of $N^2$ features at every communication step (NSQ).
2) The BCInf constant time communication (CTC) strategy where a sub-map of at the most 5 features were transmitted at each communication step. The features to include in the transmitted sub-map were selected at transmission time to be those which had greatest amount of information not yet sent.
3) The constant time communications strategy using CI in place of BCInf.

These communication strategies have been applied successfully to many randomly generated scenarios. We report on typical results obtained when we compare these communication strategies on the same scenario. The strategies were applied, in turn, to a simulation scenario in which two vehicles moved independently of each other. To enable a fair comparison we started each simulation from the same random seed. The environment through which vehicles moved was identical in each simulation. We guaranteed that the features observed and sub-maps communicated were identical for the simulations employing the BCInf and CI communication strategies and the only differences between the simulations were due to the nuances of the different communication strategies.

The simulation used 100 features, 16 of which were chosen to be well located landmarks predefined in the vehicle maps. Landmarks were located throughout the environment to counter the accumulating vehicle stochastic process noise which comes to dominate vehicle and map estimates. Landmarks were predefined in the SLAM estimates of all vehicles. These landmarks may have been mapped by a previous recce of part of the environment. For example, outdoor landmarks which have been mapped by satellites may be visible through windows by an indoor mapping team of vehicles.

Figure 4 (j) illustrates this simulation world, marking the feature locations, landmarks and vehicle paths. A nonlinear tricycle motion model was implemented to estimate the position and orientation of each vehicle within the map as in [6]. Each vehicle was equipped with a range/bearing sensor which gave observations to features at a frequency of 1Hz. Vehicles were able to communicate map information at a frequency of 0.1Hz. They were each allowed to communicate sub-maps for at most five features each communication step.

The results of the simulation are shown in Figure 4. Although, the accuracy of the vehicle estimate is similar for all the data fusion methods, feature accuracies for the BCInf constant time strategy are significantly greater than those for the CI constant time strategy. When a landmark is observed by a vehicle both the vehicle and mapped features which are close to the landmark are accurately located through local SLAM vehicle-feature correlations. The impact that the landmark observation has on the mapped feature accuracy decreases the further the feature is away from the landmark. This has an effect on the value of the communicated sub-map local SLAM/channel cross correlation bound $S$. The closer the sub-map features to the landmark the smaller the cross correlation between the sub-map and the channel estimate. Figure 3 shows the range of sub-map cross correlation bounds $S$ calculated during the simulation.

VI. CONCLUSIONS

This paper has developed a new solution to the fusion problem of correlated random vectors when the correlation is unknown but bounded. We have demonstrated how the cross correlation bounds can be determined efficiently and locally without the need for inter-agent communication or modelling. The new technology was applied to the multi-vehicle decentralised SLAM problem. The new technology was used in place of existing technology in the constant time communication strategy proposed by Nettleton et al. and yields more efficient estimates.

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APPENDIX

We provide a proof that sub-map information \( \{ \hat{y}_{AB}, Y_{AB} \} \) received by a vehicle from its channel filter can be incorporated into the local SLAM estimate using Eqns (11) to (14). These equations are valid only if the communicated information already incorporated into the local SLAM information vector can be removed by simply subtracting the channel information vector from it. That is, irrespective of whether the local SLAM information has been updated (by performing prediction or fusion operations) or not it must always be of the form:

\[
\begin{align*}
\hat{y}_B(k | k - 1) &= P^{-1} \hat{x} + G_s \hat{y}_C(k | k - 1), \\
\hat{Y}_B(k | k - 1) &= P^{-1} + G_s Y_C(k | k - 1) G_s^T
\end{align*}
\]

where the estimate \( \hat{x} \) has been obtained through observations and communications which are independent of those obtained through channel C. We will first show that, under such independence restrictions and stationarity restrictions detailed below, Eqns (15) and Eqns (16) are preserved for local filter prediction and fusion operations. Further, under such independence restrictions it is sufficient to show that Eqn (15) holds only. We will then show that Eqn (15) holds for communication.

The restriction imposed on the system is that the mapped features are stationary. Assume without, loss of generality, that the local SLAM state vector is made up of the vehicle state vector stacked on top of the map state vector. Thus, the SLAM process model \( F \) and process noise covariance matrix \( Q \) are of the form:

\[
F = \begin{bmatrix} f & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}
\]

where \( f \) is the vehicle process model, \( q \) is the vehicle process noise covariance matrix and \( I \) is the identity matrix.

**Fusion Case:** This case includes the fusion of independent estimates, \( \hat{y}_B \) say, obtained by the local SLAM filter either by observation or communication not through the channel C. The information vector obtained by fusing the new information into the existing SLAM estimate is:

\[
\hat{y}_B(k | k - 1) + \hat{y}_B = \begin{bmatrix} P^{-1} \hat{x} + G_s \hat{y}_C(k | k - 1) \end{bmatrix} + \hat{y}_B
\]

using Eqn (15)

\[
= \begin{bmatrix} P^{-1} \hat{x} + \hat{y}_B \end{bmatrix} + G_s \hat{y}_C(k | k - 1).
\]

**Stochastic Prediction Case:** Before proceeding we will simplify the notation. For readability define the local SLAM information vector \( P^{-1} \hat{x}_1 = \hat{y}_B(k | k - 1) \) and corresponding information matrix \( P^{-1} = Y_B(k | k - 1) \). Also define the channel contribution to the SLAM information vector \( X^{-1} \hat{z} = G_s \hat{y}_C(k | k - 1) \) and its information matrix \( X^{-1} = G_s Y_C(k | k - 1) G_s^T \). Eqns (15) and (16) become:

\[
\begin{align*}
P^{-1} \hat{x}_1 &= P^{-1} \hat{x} + X^{-1} \hat{z}, \quad (18) \\
P^{-1} &= P^{-1} + X^{-1}. \quad (19)
\end{align*}
\]

Let \( \hat{x}_2 \) and \( P_2 \) be the local SLAM state estimate and covariance matrix respectively after the SLAM filter prediction step:

\[
\hat{x}_2 = F \hat{x}_1 \quad \text{and} \quad P_2 = FP_1 FT + Q. \quad (20)
\]

Firstly, it is straightforward to show that \( X^{-1}Q = QX^{-1} = 0 \) and assuming that the process model \( F \) is invertible:

\[
X^{-1}(FT)^{-1} = (FT)^{-1}X^{-1} = X^{-1}F = X^{-1}. \quad (21)
\]

Further:

\[
\begin{align*}
P_2 &= F[FP^{-1} + X^{-1}]^{-1}FT + Q \quad \text{using (19) and (20)} \\
&= [(FPFT)^{-1} + X^{-1}]^{-1} + Q \quad \text{using (21)} \\
&= [(FPFT)^{-1} + X^{-1}]^{-1}[1 + ((FPFT)^{-1} + X^{-1})Q] \\
&= [(FPFT)^{-1} + X^{-1}]^{-1}[1 + (FPFT)^{-1}Q] \\
&\quad \text{using } X^{-1}Q = 0
\end{align*}
\]

Therefore,

\[
\begin{align*}
g_2 &= P_2^{-1} \hat{x}_2 \\
&= P_2^{-1} F \hat{x}_1 \quad \text{using (20)} \\
&= [1 + (FPFT)^{-1}Q]^{-1}[(FPFT)^{-1} + X^{-1}]F \hat{x}_1 \\
&= [1 + (FPFT)^{-1}Q]^{-1}(FT)^{-1}[P^{-1} + X^{-1}]F \hat{x}_1 \quad \text{using (21)} \\
&= [1 + (FPFT)^{-1}Q]^{-1}(FT)^{-1}[P^{-1} \hat{x} + X^{-1} \hat{z}] \quad \text{using (18) and (19)} \\
&= [1 + (FPFT)^{-1}Q]^{-1}[(FPFT)^{-1}F \hat{x} + X^{-1} \hat{z}] \quad \text{using (21)} \\
&= [FPFT + Q]^{-1}[FPFT + Q]^{-1}[1 - (1 + (FPFT)^{-1}Q)]X^{-1} \hat{z} + X^{-1} \hat{z} \times \text{using } X^{-1}Q = 0 \\
&= [FPFT + Q]^{-1}F \hat{x} + X^{-1} \hat{z} \text{ using } X^{-1}Q = 0 \\
&= [FPFT + Q]^{-1}F \hat{x} + G_s \hat{y}_C(k | k - 1).
\end{align*}
\]
Communication Case: We will show that Eqn (15) holds for inter-vehicle communication. First, we will show the following intuitively appealing fact, that the information in the vehicle SLAM estimate always exceeds the shared information encoded in the channel filter. That is, for all $k$:

$$Y_B(k | k - 1) \geq G_s Y_C(k | k - 1) G_s^T , \quad (22)$$

$$Y_B(k | k) \geq G_s Y_C(k | k) G_s^T . \quad (23)$$

We have shown above that these inequalities are preserved for prediction and fusion by vehicle B’s local SLAM filter. It is sufficient to show that this inequality holds for two further cases: after vehicle B has communicated to vehicle A and after A has communicated to B. For the case where B communicates to A assume that, immediately prior to communication, vehicle B’s information exceeds that of the channel:

$$G_s Y_C(k | k - 1) G_s^T \leq Y_B(k | k - 1) \leq Y_B(k | k) .$$

Vehicle B communicates its sub-map $Y_{BC}(k | k) = [G_m P_B(k | k) G_m]^T$ to its channel filter where $G_m$ is an appropriate selection matrix. We note that $Y_{BC}(k | k) \leq Y_B(k | k)$. The channel then fuses vehicle B’s communicated sub-map using BCInf. Using the fact that $G_s G_s^T Y_{BC}(k | k) G_s^T = Y_{BC}(k | k)$ we get:

$$G_s Y_C(k | k) G_s^T = G_s \left[ Y_C(k | k - 1) + G_s^T Y_{BC}(k | k) G_s \right] G_s^T$$

$$\leq Y_B(k | k) + G_s^T Y_{BC}(k | k) G_s G_s^T$$

$$\leq Y_B(k | k) .$$

For the case where A communicates to B assume that immediately prior to communication vehicle B’s information exceeds that of the channel:

$$G_s Y_C(k | k - 1) G_s^T \leq Y_B(k | k - 1) .$$

Immediately after vehicle B fuses the new sub-map communicated to it by the channel, we get, using Eqs (11) to (14) and Eqn (22):

$$Y_B(k | k) = Y_B(k | k - 1) + G_s Y_C(k | k - 1) - Y_C(k | k - 1) G_s^T$$

$$\geq Y_B(k | k - 1) - G_s Y_C(k | k - 1) G_s + G_s Y_C(k | k) G_s^T$$

Thus, we have shown that the inequalities (22) and (23) hold for all $k$.

If $Y_B(k | k - 1) \geq G_s Y_C(k | k - 1) G_s^T$ then $Y_B(k | k - 1) = G_s Y_C(k | k - 1) G_s^T + Y$ where $Y = Y_B(k | k - 1) - G_s Y_C(k | k - 1) G_s^T \geq 0$. Thus, there exists a random vector $\hat{y}$ with covariance $Y$ such that:

$$\hat{y}_B(k | k - 1) = \hat{y}_C(k | k - 1) + \hat{y} \quad (24)$$

and $G_s \hat{y}_C(k | k - 1)$ and $\hat{y}$ are uncorrelated. Thus, $\hat{y}_B(k | k - 1)$ is of the appropriate form specified by Eqn (15) and this completes the proof. We note that the reasoning above applies to vehicle A also. Thus, we can derive a similar expression to Eqn (24) for vehicle A:

$$\hat{y}_A(k | k - 1) = \hat{y}_C(k | k - 1) + \hat{y}' \quad (25)$$

where $\hat{y}'$ is orthogonal to $\hat{y}_C(k | k - 1)$. We have used our assumption, as stipulated in Section IV, that the channel and vehicle A’s local SLAM mapped features are similarly ordered.

Finally, we prove Eqn (10). The sub-map that vehicle A has chosen to communicate to the channel is:

$$\hat{x}_{AC}(k | k) = G_m \hat{x}_A(k | k) .$$

We also have, from Eqn (25):

$$\hat{y}_A(k | k - 1) = \hat{y}_C(k | k - 1) + \hat{y}' .$$

Any new observations that vehicle A makes at time $k$ are uncorrelated with any previous observations and are orthogonal to $\hat{y}_C(k | k - 1)$. Thus,

$$\hat{y}_A(k | k) = \hat{y}_C(k | k - 1) + \hat{y}''$$

where $\hat{y}''$ is some vector orthogonal to $\hat{y}_C(k | k - 1)$. Therefore,

$$\text{Cov}[G_m^T \hat{x}_{AC}(k | k), \hat{x}_C(k | k - 1)]$$

$$= G_m^T G_m \text{Cov}[\hat{x}_A(k | k), \hat{x}_C(k | k - 1)]$$

$$= G_m^T G_m Y_A^{-1}(k | k) Y_C(k | k - 1) Y_C(k | k - 1)^{-1} .$$
Fig. 4. Simulation Results. The error and $2\sigma$ bounds for a vehicle state are given for the (a) NSQ, (b) BCInf constant time and (c) CI constant time strategies. The error and $2\sigma$ bounds for a typical feature are also given for the (d) NSQ, (e) BCInf constant time and (f) CI constant time strategies. The total uncertainty encoded in the SLAM estimates over all vehicles is expressed as the determinant of the SLAM covariance matrix and is given for (g) NSQ, (h) BCInf constant time and (i) CI constant time (note the log scale). The simulation world is shown in (j) and both vehicle paths are shown. Also shown are the features (as dots) and landmarks (as crosses). For clarity the $8\sigma$ covariance ellipses (solid and dashed lines distinguishing the vehicles) for features and landmarks are given when the vehicles reach the end of the mapping run. Shown are estimate ellipses for the (k) BCInf constant time and (l) CI constant time strategies.