How to Use Decision Theory to Choose Among Mechanisms

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Point Predictions and Policy (mechanism) Choices *should* make use of decision theory.

- Need a probability distribution over states of the world (i.e. behavior of players).
- Set-valued solution concepts (NE, BNE, QRE, etc.) do not provide this.
- Predictive Game Theory (PGT) models do provide this.
PGT - the entire class of statistical game theoretic models (Wolpert and Bono).

Today: a Bayesian PGT model.

- Uses Quantal Response Equilibrium (QRE) (McKelvey and Palfrey) related concepts.
- Other models use: epsilon equilibrium (Radner), level-k (Costa-Gomes and Crawford), “intelligence” (Wolpert)
Bayesian PGT Model

- Behavioral Profiles: \( q \) (mixed strategy profiles i.e. product distributions \( q \in \times_i \Delta(X_i) \))
- Prior: \( P(q) \)
- Data: \( \mathcal{I} \)
- Likelihood: \( L(\mathcal{I}|q) \)
- Posterior:

\[
P(q|\mathcal{I}) \propto P(q)L(\mathcal{I}|q)
\]
Quantifies researcher’s beliefs about the relative likelihoods of behavioral profiles without regard to the data, $\mathcal{J}$. There are many ways to do this...

**Entropic Prior**

\[ P(q) \propto \exp (\delta S(q)) \]

where \( S(q) = -\sum_i \sum_{x_i} q_i(x_i) \ln(q_i(x_i)) \) is the Shannon’s Entropy of \( q \) and \( \delta \) is a real-valued parameter.
Reports the likelihood of $\mathcal{I}$ given the behavioral profile $q$. There are many ways to do this...

**QR-rationality**

$U_{q- i}^i$: the vector of expected payoffs to each of $i$’s pure strategies given $q_{- i}$.

Logit quantal response of $i$ to $q_{- i}$ is:

$$L_{U_{q- i}^i, \beta_i}(x_i) \propto \exp[\beta_i E_q(u_i|x_i)]$$

Given a $q$, we find the parameter $\beta_i$ that minimizes the Kullback-Leibler divergence of $L_{U_{q- i}^i, \beta_i}$ from $q_i$:

$$D(q_i||L_{U_{q- i}^i, \beta_i}) = \sum_{x_i} q_i(x_i) \ln \left( \frac{q_i(x_i)}{L_{U_{q- i}^i, \beta_i}(x_i)} \right)$$
Likelihood

QR-rationality

Minimizing $D(q_i||\mathcal{L}_{U_{q-i}})$ for each player yields $\beta^*(q)$, where $\beta^*_i(q)$ is a quantification of $i$’s rationality when faced with $q_{-i}$.

Here we’ll use:

$$\mathcal{L}(\mathcal{I}|q) \propto \prod_i [\tanh(\beta_i(q)) + 1]^{\alpha_i}$$

The greater the rationality, the greater the likelihood.
3 Different Cournot Games

5 Equilibria

Asymmetric

3 Equilibria
Elements of the point prediction problem:

1. a set of alternatives - the set of quantity profiles
2. a probability distribution over states of the world - the PGT posterior
3. an objective - a quantification of the modeler’s preferences (i.e. a loss function)
Predicting a single quantity profile with Decision Theory

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Assume quadratic loss, \( L(x, x') = \|x' - x\|^2 \), then the optimal choice is

\[
x^* = \arg\min_{x'} \int_{q,x} q(x)\|x' - x\|^2 P(q|\mathcal{I})dxdq.
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| Asymmetric   | 9.4, 8.6   | 0.55, 16.8   | 6.3, 8.2 |
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<table>
<thead>
<tr>
<th>PGT ( x^* )</th>
<th>NE ( E(x) )</th>
<th>QRE ( E(x) )</th>
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note: if we want to predict \( q \)'s, then NE and QRE provide either a degenerate choice of one alternative or are unusable.
**Problem:** A policy-maker (PM) must choose low taxes, $\tau_L$ or high taxes, $\tau_H$.

- PM’s objective under tax level $m$: $w_m(q)$
- Assume: unique NE $q^{H*}$ and $q^{L*}$, players are fully rational, etc.,

⇒ PM simply compares $w_L(q^{L*})$ and $w_H(q^{H*})$.

What if there are multiple equilibria? Or we cannot strictly make the assumptions of the equilibrium concept (i.e. rationality)?

⇒ A PM cannot make a rational choice using equilibrium concepts?
Choosing Among Mechanisms with PGT and Decision Theory

For the duopoly, assume external costs $EC(x) = e_1 x$, then social welfare is:

$$w_m(q) = \mathbb{E}_q[\pi_A + \pi_B] + \mathbb{E}_q[x_A + x_B](\tau_m - e_1).$$

To choose a mechanism using a PGT model, PM selects the mechanism, $m$, that maximizes expected social welfare over the corresponding posterior.

$$\mathbb{E}[w_m(q)] = \int_q w_m(q)P_m(q|\mathcal{I})dq.$$
Comparing Mechanisms with Multiple Equilibria

Suppose $\tau_H = 4$, $\tau_L = 2$ and $e_1 = 1$.

For “3 Equilibria”:

**PGT Results**
- $E[w_0(q)] \approx -0.3$
- $E[w_L(q)] \approx 6.1$
- $E[w_H(q)] \approx 5.2$

**Equilibrium Results**
- QRE ($\beta = 0.5$) & NE:
  \[ w_0(q^{0*}) < w_L(q^{L*}) < w_H(q^{H*}) \]
Questions real-world PM/stakeholders ask:

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Much More Information than $E[w_m(q)]$

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We have all the distributional information, so we can answer all of these.
$\alpha = .75$, QRE ($\beta = .5$)
More Informed PM/Stakeholders

- $\tau_H = 4$
  - strongly bi-modal distribution
  - highest variance in social welfare
  - highest probability of welfare greater than 15, lower than -5
- $\tau_L = 2$
  - nearly first-order stochastic dominates the zero tax distribution.
  - lowest probability of social welfare less than zero
  - highest probability of social welfare greater than 10
Example Constraint: Probability that firm profits are less than zero is greater than $\epsilon$. 
Main Point

Two types of risk:

1. idiosyncratic uncertainty about outcomes \( w_m(x) \) for a given \( q \)
2. systematic uncertainty about behavior \( q \)

Even if the PM is not averse to the risk that a given behavior \( q \) will produce bad outcomes \( w_m(x) \), he may still be averse to the risk that the policy will systematically elicit bad behavior.

Suppose PM’s objective function is:

\[
g(w_m(q)) = (w_m(q))^r = (E_q[\pi_A + \pi_B] + E_q[x_A + x_B](\tau_m - e_1))^r
\]

where \( r \in [0, 1] \)

**Real World Justification:** Major market changes are the result of costly legislative processes, and are often very difficult to retract once in place.
Posterior Moments (5 Equilibria $\delta = 1, \alpha = 4$)

$$E[q(x)] = \int_{\Delta x} q(x) P(q|\mathcal{I}) dq$$
Modeler Uncertainty about Utilities

Let $\mathcal{I} = \{\mathcal{I}' = 5 \text{ Equilibria}, \mathcal{I}'' = \text{Asymmetric}\}$. Modeler believes with probability $k$ that $\mathcal{I}'$ is true, and with probability $1 - k$ that $\mathcal{I}''$. Then $\mathcal{L}(\mathcal{I} | q) = k\mathcal{L}(\mathcal{I}' | q) + (1 - k)\mathcal{L}(\mathcal{I}'' | q)$. 

![3D graph showing density function](image)
Summary

- Point Predictions and Mechanism Choices *should* utilize decision theory.
- Need models that produce distributions over behavioral profiles
- This leads to:
  1. rational choices by PM
  2. ability to incorporate constraints
  3. incorporating risk information
  4. making modeler uncertainty explicit
  5. dealing with multiple equilibria

- Future Work:
  1. more/better PGT models
  2. include experimental/demographic data
  4. Predictive models for bargaining, cooperative, etc.
Thank you!
Parameters

- Best response functions for $\bar{x}_i = 20; d_{i1} = 20.4; d_{i2} = 2.165; d_{i3} = 0.12; d_{i4} = 0.0025; c_{i1} = 16,000,000$ for $i = A, B$.

- Best response functions for the same parameters as in figure 8, except that $d_{A1} = 19.1$ instead of 20.4.

- Best response functions for $\bar{x}_i = 9; d_{i1} = 7.1; d_{i2} = 0.8; d_{i3} = 0.15; d_{i4} = 0.0125; c_{i1} = 401.7$ for $i = A, B$. 