SIGNAL PROCESSING

B14 Option – 4 lectures

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Recommended texts

- Oppenhein & Shafer. Digital signal processing. Prentice Hall
- Matlab Signal processing toolbox manual.
Lecture 1 - Introduction

1.1 Introduction

Signal processing is the treatment of signals (information-bearing waveforms or data) so as to extract the wanted information, removing unwanted signals and noise. There are compelling applications as diverse as the analysis of seismic waveforms or the recording of moving rotor blade pressures and heat transfer rates.

In these lectures we will consider just the very basics, mainly using filtering as a case example for signal processing.
1.1.1 Analogue vs. Digital Signal Processing

Most of the signal processing techniques mentioned above could be used to process the original analogue (continuous-time) signals or their digital version (the signals are sampled in order to convert them to sequences of numbers). For example, the earliest type of “voice coder” developed was the channel vocoder which consists mainly of a bank of band-pass filters. The trend is, therefore, towards digital signal processing systems; even the well-established radio receiver has come under threat. The other great advantage of digital signal processing lies in the ease with which non-linear processing may be performed. Almost all recent developments in modern signal processing are in the digital domain.

This lecture course will start by covering the basics of the design of simple (analogue) filters as this a pre-requisite to understanding much of the underlying theory of digital signal processing and filtering.
1.2 Summary/Revision of basic definitions

1.2.1 Linear Systems

A linear system may be defined as one which obeys the **Principle of Superposition**. If \( x_1(t) \) and \( x_2(t) \) are inputs to a linear system which gives rise to outputs \( y_1(t) \) and \( y_2(t) \) respectively, then the combined input \( ax_1(t) + bx_2(t) \) will give rise to an output \( ay_1(t) + by_2(t) \), where \( a \) and \( b \) are arbitrary constants.

**Notes**

- If we represent an input signal by some support in a frequency domain, \( \mathcal{F}_{in} \) (i.e. the set of frequencies present in the input) then no new frequency support will be required to model the output, i.e.

\[
\mathcal{F}_{out} \subseteq \mathcal{F}_{in}
\]

- Linear systems can be broken down into simpler sub-systems which can be re-arranged in any order, i.e.

\[
x \rightarrow g_1 \rightarrow g_2 \equiv x \rightarrow g_2 \rightarrow g_1 \equiv x \rightarrow g_{1,2}
\]
1.2.2 Time Invariance

A time-invariant system is one whose properties do not vary with time (i.e. the input signals are treated the same way regardless of their time of arrival); for example, with discrete systems, if an input sequence $x(n)$ produces an output sequence $y(n)$, then the input sequence $x(n - n_o)$ will produce the output sequence $y(n - n_o)$ for all $n_o$.

1.2.3 Linear Time-Invariant (LTI) Systems

Most of the lecture course will focus on the design and analysis of systems which are both linear and time-invariant. The basic linear time-invariant operation is “filtering” in its widest sense.

1.2.4 Causality

In a causal (or realisable) system, the present output signal depends only upon present and previous values of the input. (Although all practical engineering systems are necessarily causal, there are several important systems which are non-causal (non-realisable), e.g. the ideal digital differentiator.)
1.2.5  Stability

A stable system (over a finite interval $T$) is one which produces a bounded output in response to a bounded input (over $T$).

1.3  Linear Processes

Some of the common signal processing functions are amplification (or attenuation), mixing (the addition of two or more signal waveforms) or un-mixing\(^1\) and filtering. Each of these can be represented by a linear time-invariant “block” with an input-output characteristic which can be defined by:

- The impulse response $g(t)$ in the time domain.

- The transfer function in a frequency domain. We will see that the choice of frequency basis may be subtly different from time to time.

As we will see, there is (for the systems we examine in this course) an invertible mapping between the time and frequency domain representations.

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\(^1\)This linear unmixing turns out to be one of the most interesting current topics in signal processing.
1.4 Time-Domain Analysis – convolution

Convolution allows the evaluation of the output signal from a LTI system, given its impulse response and input signal.

The input signal can be considered as being composed of a succession of impulse functions, each of which generates a weighted version of the impulse response at the output, as shown in 1.1. The output at time \( t \), \( y(t) \), is obtained simply by adding the effect of each separate impulse function – this gives rise to the convolution integral:

\[
y(t) = \sum_{\tau} \{x(t - \tau)d\tau \} g(\tau) \xrightarrow{d\tau \rightarrow 0} \int_{0}^{\infty} x(t - \tau)g(\tau)d\tau
\]

\( \tau \) is a dummy variable which represents time measured “back into the past” from the instant \( t \) at which the output \( y(t) \) is to be calculated.
Figure 1.1: Convolution as a summation over shifted impulse responses.
1.4.1 Notes

- Convolution is commutative. Thus:
  \[ y(t) \text{ is also } \int_{0}^{\infty} x(\tau)g(t - \tau)d\tau \]

- For discrete systems convolution is a summation operation:
  \[ y[n] = \sum_{k=0}^{\infty} x[k]g[n - k] = \sum_{k=0}^{\infty} x[n - k]g[k] \]

- **Relationship between convolution and correlation** The general form of the convolution integral
  \[ f(t) = \int_{-\infty}^{\infty} x(\tau)g(t - \tau)d\tau \]

  is very similar\(^2\) to that of the cross-correlation function relating 2 variables \( x(t) \) and \( y(t) \)
  \[ R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) \cdot y(t - \tau)dt \]

  *Convolution* is hence an integral over *lags* at a fixed *time* whereas *correlation* is the integral over *time* for a fixed *lag*.

\(^2\)Note that the lower limit of the integral can be \(-\infty\) or 0. Why?
- **Step response** The step function is the time integral of an impulse. As integration (and differentiation) are *linear* operations, so the order of application in a LTI system does not matter:

\[
\delta(t) \rightarrow \int dt \rightarrow g(t) \rightarrow \text{step response}
\]

\[
\delta(t) \rightarrow g(t) \rightarrow \int dt \rightarrow \text{step response}
\]
1.5 Frequency-Domain Analysis

LTI systems, by definition, may be represented (in the continuous case) by linear differential equations (in the discrete case by linear difference equations). Consider the application of the linear differential operator, $D$, to the function $f(t) = e^{st}$:

$$Df(t) = sf(t)$$

An equation of this form means that $f(t)$ is the eigenfunction of $D$. Just like the eigen analysis you know from matrix theory, this means that $f(t)$ and any linear operation on $f(t)$ may be represented using a set of functions of exponential form, and that this function may be chosen to be orthogonal. This naturally gives rise to the use of the Laplace and Fourier representations.
The Laplace transform:

\[ X(s) \longrightarrow \text{Transfer function } G(s) \longrightarrow Y(s) \]

where,

\[ X(s) = \int_{0}^{\infty} x(t)e^{-st}dt \quad \text{Laplace transform of } x(t) \]

\[ Y(s) = G(s)X(s) \]

where \( G(s) \) can be expressed as a pole-zero representation of the form:

\[ G(s) = \frac{A(s - z_1) \ldots (s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_n)} \]

(NB: The inverse transformation, ie. obtaining \( y(t) \) from \( Y(s) \), is not a straightforward mathematical operation.)
• The Fourier transform:

\[ X(j\omega) \rightarrow \text{Frequency response } G(j\omega)Y(j\omega) \]

where,

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \quad \text{Fourier transform of } x(t) \]

and

\[ Y(j\omega) = G(j\omega)X(j\omega) \]

The output time function can be obtained by taking the inverse Fourier transform:

\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t} \, d\omega \]
1.5.1 Relationship between time & frequency domains

**Theorem**
If $g(t)$ is the impulse response of an LTI system, $G(j\omega)$, the Fourier transform of $g(t)$, is the frequency response of the system.

**Proof**
Consider an input $x(t) = A\cos \omega t$ to an LTI system. Let $g(t)$ be the impulse response, with a Fourier transform $G(j\omega)$.

Using convolution, the output $y(t)$ is given by:

$$y(t) = \int_{0}^{\infty} A\cos(\omega(t-\tau))g(\tau)\,d\tau$$

$$= \frac{A}{2} \int_{0}^{\infty} e^{j\omega(t-\tau)}g(\tau)\,d\tau + \frac{A}{2} \int_{0}^{\infty} e^{-j\omega(t-\tau)}g(\tau)\,d\tau$$

$$= \frac{A}{2} e^{j\omega t} \int_{-\infty}^{\infty} g(\tau)e^{-j\omega \tau}\,d\tau + \frac{A}{2} e^{-j\omega t} \int_{-\infty}^{\infty} g(\tau)e^{j\omega \tau}\,d\tau$$

(lower limit of integration can be changed from 0 to $-\infty$ since $g(\tau) = 0$ for
\[ t < 0 \]

\[
A \frac{1}{2} \{ e^{j\omega t} G(j\omega) + e^{-j\omega t} G(-j\omega) \}
\]

Let \( G(j\omega) = Ce^{j\phi} \) ie. \( C = |G(j\omega)|, \quad \phi = \arg\{G(j\omega)\} \)

Then \( y(t) = \frac{AC}{2} \{ e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)} \} = CA \cos(\omega t + \phi) \)

i.e. an input sinusoid has its amplitude scaled by \( |G(j\omega)| \) and its phase changed by \( \arg\{G(j\omega)\} \), where \( G(j\omega) \) is the Fourier transform of the impulse response \( g(t) \).
Theorem
Convolution in the time domain is equivalent to multiplication in the frequency domain i.e.

\[ y(t) = g(t) * x(t) \equiv F^{-1}\{Y(j\omega) = G(j\omega)X(j\omega)\} \]

and

\[ y(t) = g(t) * x(t) \equiv L^{-1}\{Y(s) = G(s)X(s)\} \]

Proof
Consider the general integral (Laplace) transform of a shifted function:

\[
L\{f(t - \tau)\} = \int_{t} f(t - \tau)e^{-st}dt \\
= e^{-s\tau}L\{f(t)\}
\]

Now consider the Laplace transform of the convolution integral

\[
L\{f(t) * g(t)\} = \int_{t} \int_{\tau} f(t - \tau)g(\tau)d\tau e^{-st}dt \\
= \int_{\tau} g(\tau)e^{-s\tau}d\tau L\{f(t)\} \\
= L\{g(t)\}L\{f(t)\}
\]

By allowing \( s \rightarrow j\omega \) we prove the result for the Fourier transform as well.
1.6 Filters

**Low-pass**: to extract short-term average or to eliminate high-frequency fluctuations (e.g. noise filtering, demodulation, etc.)

**High-pass**: to follow small-amplitude high-frequency perturbations in presence of much larger slowly-varying component (e.g. recording the electrocardiogram in the presence of a strong breathing signal)

**Band-pass**: to select a required modulated carrier frequency out of many (e.g. radio)

**Band-stop**: to eliminate single-frequency (e.g. mains) interference (also known as notch filtering)
Figure 1.2: Standard filters.
1.7 Design of Analogue Filters

We will start with an analysis of analogue low-pass filters, since a low-pass filter can be mathematically transformed into any other standard type.

Design of a filter may start from consideration of

- The desired frequency response.
- The desired phase response.

The majority of the time we will consider the first case. Consider some desired response, in the general form of the (squared) magnitude of the transfer function, i.e. $|G(s)|^2$. This response is given as

$$|G(s)|^2 = G(s)G^*(s)$$

where $*$ denotes complex conjugation. If $G(s)$ represents a stable filter (its poles are on the LHS of the s-plane) then $G^*(s)$ is unstable (as its poles will be on the RHS).

The design procedure consists then of

- Considering some desired response $|G(s)|^2$ as a polynomial in even powers of $s$. 

• Designing the filter with the stable part of \( G(s), G^*(s) \).

This means that, for any given filter response in the positive frequency domain, a mirror image exists in the negative frequency domain.

1.7.1 Ideal low-pass filter

![Diagram of the ideal low-pass filter](image)

Figure 1.3: The ideal low-pass filter. Note the requirement of response in the negative frequency domain.

Any frequency-selective filter may be described either by its frequency response
(more common) or by its impulse response. The narrower the band of frequencies transmitted by a filter, the more extended in time is its impulse response waveform. Indeed, if the support in the frequency domain is decreased by a factor of a (i.e. made narrower) then the required support in the time domain is increased by a factor of a (you should be able to prove this).

Consider an ideal low-pass filter with a “brick wall” amplitude cut-off and no phase shift, as shown in Fig. 1.3.

Calculate the impulse response as the inverse Fourier transform of the frequency response:

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega t} d\omega = \frac{1}{2\pi jt} (e^{j\omega_c t} - e^{-j\omega_c t})$$

hence,

$$g(t) = \frac{\omega_c}{\pi} \left( \frac{\sin \omega_c t}{\omega_c t} \right)$$

Figure 1.4 shows the impulse response for the filter (this is also referred to as the filter kernel).

The output starts infinitely long before the impulse occurs – i.e. the filter is not realisable in real time.
Figure 1.4: Impulse response (filter kernel) for the ILPF. The zero crossings occur at integer multiples of $\pi/\omega_c$.

A delay of time $T$ such that

$$g(t) = \frac{\omega_c \sin \omega_c (t - T)}{\pi} \omega_c (t - T)$$

would ensure that most of the response occurred after the input (for large $T$). The use of such a delay, however, introduces a phase lag proportional to frequency, since $\arg\{G(j\omega)\} = \omega T$. Even then, the filter is still not exactly realisable; instead the design of analogue filters involves the choice of the most suitable approximation to the ideal frequency response.
1.8 Practical Low-Pass Filters

Assume that the low-pass filter transfer function \( G(s) \) is a rational function in \( s \). The type of filter to be considered in the next few pages is *all-pole design*, which means that \( G(s) \) will be of the form:

\[
G(s) = \frac{1}{(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0)}
\]

or

\[
G(j\omega) = \frac{1}{(a_n (j\omega)^n + a_{n-1} (j\omega)^{n-1} + \ldots + a_1 j\omega + a_0)}
\]

The *magnitude-squared response* is \(|G(j\omega)|^2 = G(j\omega) \cdot G(-j\omega)\). The denominator of \(|G(j\omega)|^2\) is hence a polynomial in even powers of \( \omega \). Hence the task of approximating the ideal magnitude-squared characteristic is that of choosing a suitable denominator polynomial in \( \omega^2 \), i.e. selecting the function \( H \) in the following expression:

\[
|G(j\omega)|^2 = \frac{1}{1 + H\{\left(\frac{\omega}{\omega_c}\right)^2\}}
\]

where \( \omega_c = \text{nominal cut-off frequency} \) and \( H = \text{rational function of } \left(\frac{\omega}{\omega_c}\right)^2 \).
The choice of $H$ is determined by functions such that $1 + H\{(\omega/\omega_c)^2\}$ is close to unity for $\omega < \omega_c$ and rises rapidly after that.

1.9 Butterworth Filters

$$H\{\left(\frac{\omega}{\omega_c}\right)^2\} = \left(\frac{\omega}{\omega_c}\right)^2^n = \left(\frac{\omega}{\omega_c}\right)^{2n}$$

i.e.

$$|G(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2n}}$$

where $n$ is the order of the filter. Figure 1.5 shows the response on linear (a) and log (b) scales for various orders $n$. 

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Figure 1.5: Butterworth filter response on (a) linear and (b) log scales. On a log-log scale the response, for $\omega > \omega_c$, falls off at approx -20db/decade.
1.9.1 Butterworth filter – notes

1. $|G| = \frac{1}{\sqrt{2}}$ for $\omega = \omega_c$ (i.e. magnitude response is 3dB down at cut-off frequency)

2. For large $n$:
   - in the region $\omega < \omega_c$, $|G(j\omega)| = 1$
   - in the region $\omega > \omega_c$, the steepness of $|G|$ is a direct function of $n$.

3. Response is known as maximally flat, because

$$\left. \frac{d^n G}{d\omega^n} \right|_{\omega=0} = 0 \text{ for } n = 1, 2, \ldots, 2N - 1$$

**Proof**

Express $|G(j\omega)|$ in a binomial expansion:

$$|G(j\omega)| = \left\{ 1 + \left(\frac{\omega}{\omega_c}\right)^{2n} \right\}^{-\frac{1}{2}} = 1 - \frac{1}{2} \left(\frac{\omega}{\omega_c}\right)^{2n} + \frac{3}{8} \left(\frac{\omega}{\omega_c}\right)^{4n} - \frac{5}{16} \left(\frac{\omega}{\omega_c}\right)^{6n} + \ldots$$

It is then easy to show that the first $2n - 1$ derivatives are all zero at the origin.
1.9.2 Transfer function of Butterworth low-pass filter

\[ |G(j\omega)| = \sqrt{G(j\omega)G(-j\omega)} \]

Since \( G(j\omega) \) is derived from \( G(s) \) using the substitution \( s \to j\omega \), the reverse operation can also be done, i.e. \( \omega \to -js \)

\[ \sqrt{G(s)G(-s)} = \frac{1}{\sqrt{1 + (-j \frac{s}{\omega_c})^{2n}}} \]

or

\[ G(s)G(-s) = \frac{1}{1 + (\frac{-js}{\omega_c})^{2n}} \]

Thus the poles of

\[ \frac{1}{1 + (\frac{-js}{\omega_c})^{2n}} \]

belong either to \( G(s) \) or \( G(-s) \). The poles are given by:

\[ \left(\frac{-js}{\omega_c}\right)^{2n} = -1 = e^{j(2k+1)\pi}, \quad k = 0, 1, 2, \ldots, 2n - 1 \]
Thus
\[-js\over \omega_c = e^{j(2k+1)\pi 2n}\]

Since \(j = e^{j\pi 2}\), then we have the final result:

\[s = \omega_c \exp j \left[ \frac{\pi}{2} + (2k + 1)\frac{\pi}{2n} \right]\]

i.e. the poles have the same modulus \(\omega_c\) and equi-spaced arguments. For example, for a fifth-order Butterworth low-pass filter (LPF), \(n = 5\):

\[\frac{\pi}{2n} = 18^\circ \to \frac{\pi}{2} + (2k + 1)\frac{\pi}{2n} = 90^\circ + (18^\circ, 54^\circ, 90^\circ, 126^\circ, \text{etc.})\]

i.e. the poles are at:

\[108^\circ, 144^\circ, 180^\circ, 216^\circ, 252^\circ, 288^\circ, 324^\circ, 360^\circ, 396^\circ, 432^\circ\]

in L.H. s-plane therefore stable in R.H.S. s-plane therefore unstable

We want to design a stable filter. Since each unstable pole is \((-1)\times\) a stable pole, we can let the stable ones be in \(G(s)\), and the unstable ones in \(G(-s)\).

Therefore the poles of \(G(s)\) are \(\omega_c e^{j108^\circ}, \omega_c e^{j144^\circ}, \omega_c e^{j180^\circ}, \omega_c e^{j216^\circ}, \omega_c e^{j252^\circ}\)

as shown in Figure 1.6.
Figure 1.6: Stable poles of 5-th order Butterworth filter.
\[ G(s) = \frac{1}{1 - \frac{s}{p_k}} = \frac{1}{(1 + (\frac{s}{\omega_c}))\{1 + 2 \cos 72^{\circ} \frac{s}{\omega_c} + (\frac{s}{\omega_c})^2\}\{1 + 2 \cos 36^{\circ} \frac{s}{\omega_c} + (\frac{s}{\omega_c})^2\}} \]

\[ G(s) = \frac{1}{1 + 3.2361 \frac{s}{\omega_c} + 5.2361(\frac{s}{\omega_c})^2 + 5.2361(\frac{s}{\omega_c})^3 + 3.2361(\frac{s}{\omega_c})^4 + (\frac{s}{\omega_c})^5} \]

Note that the coefficients are “palindromic” (read the same in reverse order) – this is true for all Butterworth filters. Poles are always on same radii, at \( \frac{\pi}{n} \) angular spacing, with “half-angles” at each end. If \( n \) is odd, one pole is real.

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<th>( a_3 )</th>
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Butterworth LPF coefficients for \( a \leq 8 \)
1.9.3 Design of Butterworth LPFs

The only design variable for Butterworth LPFs is the order of the filter \( n \). If a filter is to have an attenuation \( A \) at a frequency \( \omega_a \):

\[
|G|_{\omega_a}^2 = \frac{1}{A^2} = \frac{1}{1 + \left(\frac{\omega_a}{\omega_c}\right)^{2n}}
\]

i.e.

\[
n = \frac{\log(A^2 - 1)}{2 \log \frac{\omega_a}{\omega_c}}
\]

or since usually \( A \gg 1 \),

\[
n \approx \frac{\log A}{\log \frac{\omega_a}{\omega_c}}
\]

Butterworth design – example

Design a Butterworth LPF with at least 40 dB attenuation at a frequency of 1kHz and 3dB attenuation at \( f_c = 500\text{Hz} \).

**Answer**

40dB → \( A = 100 \); \( \omega_a = 2000\pi \) and \( \omega_c = 1000\pi \text{ rads/sec} \)
Therefore \( n \approx \frac{\log_{10} 100}{\log_{10} 2} = \frac{2}{0.301} = 6.64 \)

**Hence \( n = 7 \) meets the specification**

**Check:** Substitute \( s = j2 \) into the transfer function from the above table for \( n = 7 \)

\[
|G(j2)| = \frac{1}{|87.38 - 93.54j|}
\]

which gives \( A = 128.32 \).