The handouts are brief lecture notes, not a text book! You must come to the lectures, read textbooks, do the tutorials and Matlab exercises to understand this course.
Introduction

What is this course for? Throughout your career as engineers, you will be turning mathematical models of engineering systems into numerical results. This course provides Engineering Tools in the form of Numerical Methods to do this using Computers.

The study of these methods is called NUMERICAL ANALYSIS

- Real world engineering design and analysis problems are usually too complex to solve analytically!
- Numerical analysis provides algorithms (a sequence of operations) to solve problems numerically
- Computer packages (e.g. MATLAB, Mathematica NAG libraries, IDEAS for Computer Aided Design (CAD), FLUENT for Computational Fluid Dynamics (CFD)) have built in algorithms making it easier for engineers to solve problems – these can need handling with care.
- The Engineering Computation Laboratory allows you to investigate some of these methods using MATLAB in more depth.

Themes

Numerical analysis is not principally concerned with computer algorithm implementation and optimisation of speed of implementation of algorithms (computer science issues).

It is also, not just an infinite set of methods (recipes) that one has to just memorise.

In practice, there are a finite number of useful numerical methods. These can be compared in terms of
- Convergence rate/accuracy
- Robustness/stability/conditioning

(we are going to have to define measures for these properties before we do this of course….)}
Recommended text:
Kreyzig, E. *Advanced Engineering Mathematics*, 7th Ed. Chapters 18-20

Further reading:
Curtis, F. and Wheatley, P.O., *Applied Numerical Analysis*, Addison-Wesley

Plus many books on numerical analysis in college libraries and the departmental library. If you find a good one let me know and I'll advertise it.....

Lecture Course Contents (8 lectures):

1. Linear simultaneous equations $Ax=b$: rank; nullity; kernels and echelon form.
2. Conditioning of simultaneous equations: ill-conditioning; vector norms; matrix norms; condition number.
3. Iterative solution of simultaneous equations $Ax=b$: Jacobi and Gauss-Siedel algorithms; convergence.
5. Approximate representations of data: regression, overfitting, functional approximation, using orthonormal polynomials as a basis, Chebyshev polynomials, stable regression.
6. Computing derivatives and integrals: difference methods; trapezium method; Simpson’s rule and advanced quadrature methods.
8. Computing Solutions of Partial Differential Equations: elliptical (e.g. Laplace’s), parabolic (e.g. diffusion) and hyperbolic (e.g. wave) equations.
Linear Simultaneous Equations

Many engineering problems come as Linear simultaneous equations.

Consider the equation for Node 4 on the right. The Ys are admittances:

\[ (V_4 - V'_0)Y_{40} + (V_4 - V_3)Y_{43} + (V_4 - V_2)Y_{42} = 0 \]

For all nodes, this is a set of equations of the form:

\[ \sum_{j} (V_i - V_j)Y_{ij} = 0, \quad i = 1, 2, 3, \ldots \]

At first glance these equations appear homogeneous of the form \( \sum_{j} \beta_j x_j = 0 \).

However, on the left hand side we have an extra inhomogeneous equation defining the input:

\[ V_i = V'_i \]

the complete set can be written as a matrix equation \( Ax = b \).

In this case \( x = (V_1, \ldots, V_n)^T \), \( b = (V'_0, 0, 0, \ldots)^T \) and \( A \) is a matrix of admittances of the form \( Y_{ij} \).

Solving the matrix equation \( Ax = b \) calculates the voltages in the circuit.
Structural analysis (e.g. Computer aided design CAD) also gives rise to Linear simultaneous equations, if the components of the forces into nodes of a truss structure (such as the Sydney Harbour bridge below) are summed.

Let us look at formal ways to solve general linear simultaneous equations of this kind.

Rank and Nullity

Last year you solved \( Ax = b \) when \( A_{M \times N} \) is square, or \( M = N \). How?

i.e. number of equations = number of variables

What if this isn’t so? What if matrices are rectangular rather than square?

Two Cases:

1. Portrait \( M > N \) More equations than variables.

   Either equations are inconsistent \( \rightarrow \) no exact solution. Perhaps look for Optimal solution which most nearly satisfies the equations?

   Or there is some redundancy \( \rightarrow \) e.g. one equation is the sum of two others, and a solution may still exist!

2. Landscape \( M < N \) More variables than equations. \( \rightarrow \) Whole family of solutions. Can we incorporate other knowledge and find some optimal solution?

Need tests to determine redundancy and consistency in sets of linear simultaneous equations
Rank $r$ of a Matrix

A matrix $A_{mxN}$ is made up of $M$ row-vectors $a^T_m$, $m = 1, 2, \ldots M$.

Each row vector is the set of coefficients of left hand side of the equations $Ax = b$.

$r = \text{rank}(A)$ is defined as the number of linearly independent vectors of the set $a^T_m$ of the rows of $A$.

A vector $a_m$ is linearly dependent on $a_1, a_2, \ldots, a_{m-1}$ if there exist constants $\mu_1, \mu_2, \ldots, \mu_{m-1}$ such that

$$a_m = \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_{m-1} a_{m-1}$$

How do we determine rank?

Examples - “by-eye” approach:

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has rank 3.
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ have rank 2.

What are the determinants of these matrices? Is this relevant?

Let’s try some further examples:

What is the rank of

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 7 \\ 3 & 4 & 5 & 9 \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 7 \\ 3 & 4 & 5 & 8 \end{bmatrix}$?
Nullity of a Matrix

A complimentary measure to rank of \( A \), the number of effective constraints on \( x \) in \( Ax = b \), is the nullity.

Define \( n = \text{nullity}(A) = \text{number of degrees of freedom remaining in } x \). So:
\[
n = \text{nullity}(A_{MxN}) = N - r
\]
(remember, \( N \) is the number of columns, or variables)

What are the nullities of the examples above .......

The further examples:

(i) \[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 4 & 5 \\
\end{bmatrix}, \quad r=2, \ N=3, \quad n=N-r=3-2=1
\]

(ii) \[
\begin{bmatrix}
1 & 2 & 3 & 5 \\
2 & 3 & 4 & 7 \\
3 & 4 & 5 & 9 \\
\end{bmatrix}, \quad r=2, \ N=4, \quad n=N-r=4-2=2
\]

(iii) \[
\begin{bmatrix}
1 & 2 & 3 & 5 \\
2 & 3 & 4 & 7 \\
3 & 4 & 5 & 8 \\
\end{bmatrix}, \quad r=3, \ N=4, \quad n=N-r=4-3=1
\]
Example of an indeterminate structure

Resolving forces horizontally and vertically, plus taking moments, gives equilibrium equations, in matrix form:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
w \\
0 \\
\end{bmatrix}
\]

You can see that the rows of \( A \) are independent \( \rightarrow \) \( A \) has rank \( r = 3 \).

But number of columns = number of variables = \( N = 4 \).

So nullity \( n = 4 - 3 = 1 \). Since \( n > 0 \), the forces \( \begin{bmatrix} x_1, x_2, x_3 \end{bmatrix} \) are not uniquely determined.

Of course this is because the horizontal forces are indeterminate and can be made any magnitude, subject to the constraint in the second equation, i.e.

\[ x_1 + x_2 = 0 \]

**Nullity can be useful in determining whether engineering problems have real solutions.**

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The general solution of a linear simultaneous equations

\[ Ax = b \]
The Kernel of a Matrix

What can we do about finding the many solutions to indeterminate sets of simultaneous linear equations?

Remember the solutions to a differential equation such as $\ddot{y} + 3\dot{y} + 2y = \cos 3x$?
The full solution was one particular solution of the full equation plus a complimentary function solving $\ddot{y} + 3\dot{y} + 2y = 0$. We use a similar idea:

The general solution of $Ax = b$ has two parts:

1. A particular solution $x_0$ such that $Ax_0 = b$ and
2. A kernel of $A$, defined as a vector space $\ker(A) = \{x : Ax = 0\}$ where “:” here means “such that”

The Kernel is thus the set of many vectors for which $Ax = 0$. Note $0$ is a vector!

The general solution is then $x = x_0 + x'$, for all $x' \in \ker(A)$, where $\in$ reads “contained in”.

For example, in the indeterminate structure above, one particular solution (by inspection) is

$x_0 = \begin{bmatrix} 0 & W/2 & 0 & W/2 \end{bmatrix}$

and $\ker(A) = [\beta \ 0 \ -\beta \ 0]^T$ where $\beta \in \mathbb{R}$ is any real-valued constant.

This makes sense in physical terms, and gives a complete solution (of course, many solutions!)

$x = x_0 + x' = [\beta \ W/2 \ -\beta \ W/2]$
Echelon Form

We can’t continue to solve all these problems “by inspection”! We need a systematic, robust engineering method for computing rank, nullity, the kernel and particular solutions.

Use Gaussian elimination you used in the first year to generate the Echelon Form of the matrix:

**Gaussian elimination Example:**
Add multiples of one row to others to reduce matrix to an upper triangular form.

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 4 & 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 \\
0 & -1 & -2 \\
0 & -2 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 \\
0 & -1 & -2 \\
0 & 0 & 1
\end{bmatrix}
\]

**Note** that these row operations preserve the determinant and the rank of the matrix.

The rank is then just the number of non-zero rows (= 3).
The determinant is the product of the diagonal terms (= -1).

This looks useful!

Echelon form Example:
Apply similar row-operations to non-square matrices to determine their rank and nullity. Use our structure example again.

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{bmatrix}
\]

No zero rows, so rank \( r = 3 \). Nullity = 4 - 3 = 1 as before.

In the problem \( Ax = 0 \), or

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

There is one free variable, \( x_3 \), not on the leading terms of the echelon. so set \( x_3 = \beta \), any real number, and back solve to give

\[
x_4 = 0, \quad x_2 = 0, \quad x_1 + x_3 = 0 \quad \text{giving} \quad x' = k\text{er}(A) = \begin{bmatrix} \beta \\ 0 \\ -\beta \\ 0 \end{bmatrix} \text{ as before.}
\]
General solution to static indeterminacy problem

A slight variation on what we have done just now.

Reduce what is known as the augmented matrix to Echelon form in the same fashion as you did with Gaussian elimination:

Represent $Ax = b$ by the augmented matrix

$$[A \ | \ b] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & W \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & W \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Get a particular solution by setting free variable $x_3 = 0$ and back solve for remaining key variables:

$x_i = W/2, \ x_j = W/2, \ x_k = 0 \Rightarrow x_n = \begin{bmatrix} 0 & W/2 & 0 & W/2 \end{bmatrix}^T$

General solution is

$x = \begin{bmatrix} 0 & W/2 & 0 & W/2 \end{bmatrix}^T + [\beta \ 0 \ -\beta \ 0]^T \quad [As \ before]$
Proceed in three steps:

1. Consistency. In $[A' | b']$ $b'$ is linearly dependant on the columns of $A'$ so at least one solution exists.

2. Kernel. Identify free variables $x_3$, $x_4$ and solve $Ax = 0$ to give $\ker(A) = \begin{bmatrix} -\alpha & -\beta \\ -\alpha & \beta \\ \alpha & 0 \\ 0 & -\beta \end{bmatrix}$

3. Particular solution Set free variables $x_3 = x_4 = 0$ and solve $Ax = b$ to give $x = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$

The full solution is the sum of these:

$$
\begin{bmatrix}
1 \\
3 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
-\alpha \\
-\alpha \\
\alpha \\
0 \\
\end{bmatrix} + \begin{bmatrix}
-\beta \\
\beta \\
0 \\
\beta \\
\end{bmatrix} = \begin{bmatrix}
1-\alpha - \beta \\
3-\alpha + \beta \\
\alpha \\
\beta \\
\end{bmatrix}
$$

for any values of $\alpha$ and $\beta$.