CALCULUS II

4 Lectures

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PARTIAL DIFFERENTIATION I

Functions of More than One Variable

Many engineering applications involve functions of more than one variable.

For example, the volume of a circular cylinder of radius $r$ and height $h$ is
\[ V = \pi r^2 h. \]
The volume is thus a function of two independent variables $r$ and $h$.

The force $F$ that a flowing liquid exerts on an immersed body is a function of the relative velocity between the liquid and body ($v$), the frontal area of the body ($A$) and the density of the liquid ($\rho$). We can express this in mathematical form as
\[ F = f(v, A, \rho). \]
Again, in this example, the quantities $v$, $A$ and $\rho$ are completely independent of each other.
It is important to be able to distinguish between functions of more than one independent variable, and functions of dependent variables.

Consider for example the function:

\[ z = x^2 + y^2 \quad \text{where} \quad x = 2u \quad \text{and} \quad y = u^2. \]

We can substitute for \( x \) and \( y \) to give:

\[ z = 4u^2 + u^4. \]

Clearly \( z \) is a function solely of the independent variable \( u \).

Thus, because \( x \) and \( y \) are both functions of \( u \),
\( z \) is actually a function of only one independent variable, not two.
Functions of many variables can be difficult to visualise because they are hard to represent graphically.

Functions of two variables can be plotted in a 3-D coordinate system as a surface – and viewed as 3-D perspective or contour plots.

\[ f(x, y) = \sin(x) + \cos(y + x/2) \]
Partial Derivatives

Suppose we have a real, single-valued function \( f(x, y) \) of two independent variables \( x \) and \( y \).

The **partial derivative** of \( f \) with respect to \( x \) is defined as

\[
\left( \frac{\partial f}{\partial x} \right)_y = \lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x, y) - f(x, y)}{\delta x} \right\} .
\]

Here \( \frac{\partial f}{\partial x} \) means the partial derivative with respect to \( x \), while the subscript \( y \) means that \( y \) is held constant. Thus, the partial derivative with respect to \( x \) is really just the same as the ordinary derivative, with \( y \) treated as a constant instead of a variable.

Similarly, the partial derivative of \( f \) with respect to \( y \) is

\[
\left( \frac{\partial f}{\partial y} \right)_x = \lim_{\delta y \to 0} \left\{ \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \right\} .
\]
A convenient shorthand for partial derivatives is

\[
    f_x = \left( \frac{\partial f}{\partial x} \right)_y \quad \text{and} \quad f_y = \left( \frac{\partial f}{\partial y} \right)_x.
\]

**Example.** Find the partial derivatives of \( z = \sin x + \frac{1}{2} \cos \left( \frac{x}{2} + y \right) \).

\[
    z_x = \frac{\partial z}{\partial x} = \cos x - \frac{1}{4} \sin \left( \frac{x}{2} + y \right) \quad \text{and} \quad z_y = \frac{\partial z}{\partial y} = -\frac{1}{2} \sin \left( \frac{x}{2} + y \right).
\]

What do these mean?
The physical meanings of the partial derivatives should be fairly obvious.

Consider for example the partial derivative values at (3, 2):

\[ z_x = \cos 3 - \frac{1}{4} \sin \left( \frac{3}{2} + 2 \right) = -0.902 \quad \text{and} \quad z_y = -\frac{1}{2} \sin \left( \frac{3}{2} + 2 \right) = +0.175. \]

This means that

the surface has a slope of

-0.902 in a direction parallel to the x-axis (i.e. with y constant)

and +0.175 in a direction parallel to the y-axis.
Unless they are discontinuous functions, the first partial derivatives may be differentiated again to give higher partial derivatives.

For a function of two variables, there are four possible partial second derivatives:

\[
\begin{align*}
f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)_y, \\
f_{yy} &= \frac{\partial^2 f}{\partial y^2}, \\
f_{xy} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)_x, \\
f_{yx} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)_y.
\end{align*}
\]
So long as the second partial derivatives are continuous (and sometimes even if they are not),

it can be shown that partial differentiation is a **commutative** operation.

That is,

\[ f_{xy} = f_{yx} \quad \text{or} \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_x = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_y. \]
Example: Find the partial second derivatives of \( f(x, y) = e^{2x} \cos(y - x) \).

\[
\begin{align*}
    f_x &= e^{2x} \sin(y - x) + 2e^{2x} \cos(y - x) = e^{2x}[\sin(y - x) + 2\cos(y - x)] \\
    f_{xx} &= e^{2x}[-\cos(y - x) + 2\sin(y - x)] + 2e^{2x}[\sin(y - x) + 2\cos(y - x)] \\
    &= e^{2x}[4\sin(y - x) + 3\cos(y - x)]
\end{align*}
\]

And \( f_{yx} = e^{2x}[-\cos(y - x) - 2\sin(y - x)] \)

\[
\begin{align*}
    f_y &= -e^{2x} \sin(y - x) \\
    f_{yy} &= -e^{2x} \cos(y - x)
\end{align*}
\]

And \( f_{xy} = -e^{2x}[-\cos(y - x) - 2e^{2x} \sin(y - x)] = e^{2x}[\cos(y - x) - 2\sin(y - x)] = f_{yx} \)
The Total Derivative

The partial derivatives tell us how a function $f(x, y)$ changes when either of the variables $x$ or $y$ change. The next step is to consider how $f$ varies when both $x$ and $y$ change at the same time.

Small increments $\delta x$ and $\delta y$ result in an increment in the function value of $\delta f$ given by

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y)$$

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$

$$\delta f = \left[ \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \right] \delta x + \left[ \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \right] \delta y .$$
The terms in square brackets here are very similar to the definitions of the partial derivatives given earlier, so we can write

\[ \delta f \approx \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y. \]

As \( \delta x \) and \( \delta y \) become arbitrarily small, the terms in square brackets become equal to the partial differentials, and we get

\[ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \]

This is known as the total differential of \( f \). It can be seen that the first term here represents the change in \( f(x, y) \) due to a small increment in \( x \), keeping \( y \) constant, and the second term represents the change due to a small increment in \( y \), keeping \( x \) constant. The total differential is simply the sum of these two effects.

The definition above can be extended to a function of any number of independent variables \( x_1, x_2, \ldots, x_n \):
\[ df = \frac{\partial f}{\partial x_1} \, dx_1 + \frac{\partial f}{\partial x_2} \, dx_2 + \ldots + \frac{\partial f}{\partial x_n} \, dx_n . \]

Returning to the case of a function of two variables \( f(x, y) \), we will now consider some special cases where the two variables are not independent. First, suppose \( x \) and \( y \) are both differentiable functions of some other variable \( u \), i.e. \( x = x(u) \), \( y = y(u) \).

Then, dividing through the total differential by \( du \) gives:

\[ \frac{df}{du} = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy . \]  

(2)

This expression is known as the total derivative of \( f \) with respect to \( u \).

Note that, since both \( x \) and \( y \) are functions of \( u \), \( f \) can be expressed as a function of \( u \) alone. Hence \( df/du \) is an ordinary derivative, not a partial one.

We could, of course, evaluate \( df/du \) by substituting for \( x \) and \( y \) and then performing an ordinary differentiation with respect to \( u \), but often it is easier to use the partial differential approach.
Example: \( f = (1 + y^2)\sin^2 x \quad x = \tan^{-1} u \quad y = a^u \).

\[
f_x = 2\sin x \cos x \cdot (1 + y^2)
\]

\[
f_y = 2y \sin^2 x
\]

\[
\frac{dx}{du} = \frac{1}{1 + u^2}
\]

\[
y = a^u \quad \to \quad \ln y = u \ln a \quad \to \quad \frac{1}{y} \frac{dy}{du} = \ln a \quad \to \quad \frac{dy}{du} = y \ln a = a^u \ln a.
\]

\[
\therefore \quad \frac{df}{du} = \frac{2 \sin x \cos x (1 + y^2)}{1 + u^2} + 2y \sin^2 x \cdot a^u \ln a.
\]

NB: An alternative approach would be to write \( f(u) = (1 + a^{2u}) \sin^2 (\tan^{-1} u) \), then differentiate with respect to \( u \) – very awkward!
Another special case is where $y$ is itself a function of $x$, i.e. $y = y(x)$. In this case, dividing through the total differential by $dx$ gives:

\[ \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} . \]

Again, this formula may sometimes give a simpler solution than trying to evaluate $df/dx$ directly.
Lastly, in this section, we define the **exact differential** of a function. Suppose we have an expression of the form

\[ P(x, y)dx + Q(x, y)dy \]

There is said to be an exact differential if there exists a function \( f(x, y) \) such that

\[ df = P(x, y)dx + Q(x, y)dy \]

By comparing this expression to the total differential, we see that this implies:

\[
\frac{\partial f}{\partial x} = P(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = Q(x, y) .
\]

Hence the function \( f \) can be found by integration as follows:-

Evaluate both \( f = \int P(x, y)dx \) and \( f = \int Q(x, y)dy \), and compare the results to leave an expression for \( f \) involving a constant of integration.
Example: If \((x + 3y)dx + (3x + 2y)dy\) is the exact differential of \(f(x, y)\), find \(f\).

\[ x + 3y = \frac{\partial f}{\partial x} \quad \text{and} \quad 3x + 2y = \frac{\partial f}{\partial y} \]

\[ f = \frac{x^2}{2} + 3xy + Y(y) + C \]

\[ \frac{\partial f}{\partial y} = 3x + \frac{dY}{dy} = 3x + 2y \]

\[ \therefore \quad Y = y^2 \quad \text{and} \quad f = \frac{x^2}{2} + 3xy + y^2 + C \]
A Simple Engineering Application

The diagram shows a cylindrical bar of initial length $l$ and radius $r$, subjected to some forces. The volume of the cylinder is a function of the two independent variables $l$ and $r$:

$$V = \pi r^2 l$$
Now the forces will cause the bar to deform. We can calculate the linear deformations in each direction from the elastic properties of the bar, but how are these related to the change in volume of the bar?

First, we define the **strain** in a given direction to be the change in length divided by the original length. Thus we have:

Longitudinal strain: \[ \varepsilon_l = \frac{dl}{l}, \]

Radial strain: \[ \varepsilon_r = \frac{dr}{r}. \]

Now, returning to our expression for the volume, we first take logs: \[ \ln V = \ln \pi + 2\ln r + \ln l. \]
and then form the total differential of \( \ln V \):

\[
d(ln V) = \frac{\partial (ln V)}{\partial r} dr + \frac{\partial (ln V)}{\partial l} dl
\]

\[
\frac{d(ln V)}{dV} dV = \frac{2}{r} dr + \frac{1}{l} dl
\]

\[
\frac{dV}{V} = 2\epsilon_r + \epsilon_l .
\]

Thus, the proportionate change in volume, or \textbf{volumetric strain}, is simply the sum of the direct strains in three perpendicular directions.